2020

ALGEBRA I TOPICS USING GEOGEBRA

Matthew Rancourt

ALGEBRA I TOPICS USING GEOGEBRA

An Essay Submitted to the
Office of Graduate Studies
College of Arts & Sciences of
John Carroll University
In Partial Fulfillment of the Requirements
For the Degree of
Master of Arts

By
Matthew E. Rancourt
2020

Table of Contents

Introduction

Linear functions and their properties are fundamental in any Algebra I course. In my experiences teaching Algebra I, I have noticed that many students struggle with understanding some of the concepts. Students often memorize the vocabulary and can figure out how to set up graphs or solve equations, but oftentimes they do not display a true understanding. I have tried many methods to solidify a deep cognitive understanding of linear functions and their properties in my students; however I always felt that a percentage of my students were being shortchanged.

Once I discovered and began investigating the dynamic geometry platform GeoGebra, I decided to change the approach in my classes. Many of my students are visual learners and all too frequently, Algebra I textbooks are equation and definition heavy. GeoGebra allows students to investigate Algebra in a more visual and interactive way. For example, memorizing the definition of slope has a far less impact on a student than actually investigating properties of slope in a dynamic environment. I designed the lessons in this essay to build understanding of topics in Algebra I through interactive and visual methods using GeoGebra. Today's students are growing up in a technology-based society and GeoGebra is well suited for building cognitive understanding of an otherwise bland topic.

The first four lessons in this essay focus on the slope and y-intercept of a line as well as the slope-intercept form of the equation of a line, through activities and constructions in GeoGebra. Lesson 1 focuses solely on lines with positive slope. Lesson 2 expands to an investigation of lines with any slope. In Lesson 3, students work with the slope of a line between two points by computing the rise and run between two points. As the students work through these lessons, they will use GeoGebra to visually investigate how slope and y-intercept interact on a graph. Lesson 4 has students use the slope-intercept form of a line to solve word problems that they model in GeoGebra.

The next three lessons focus on operations of functions. In Lesson 5, students use GeoGebra to investigate addition and subtraction of functions. Lesson 6 expands the list of function operations, and students investigate scalar multiplication and the product of functions. Then in Lesson 7, the students compose functions. In each of these three lessons, the students use an interactive worksheet in GeoGebra with guided instructions to build understanding. The GeoGebra files for these lesson are included with the electronic version of this essay. The students then build their own GeoGebra worksheets in each of the lessons.

The final lesson in this essay, Lesson 8, provides challenge and enrichment to students. In this lesson the students will explore different configurations of lines that are coincident, parallel or intersecting. Students will use a premade interactive GeoGebra worksheet to investigate these problems. The GeoGebra file for this lesson is included with the electronic version of this essay.

Each section of this essay includes a summary of the lesson, an estimated time of completion, a list of materials, a blank student worksheet and an annotated worksheet for teachers. The lessons are designed for students to complete in groups of three or four; however, they can be modified for pairs or even individual students. When I have students work in groups or teams, each student has a specific role such as Team Leader, Resource Manager, Reporter and Facilitator. Of course, teachers can modify these lessons to accommodate their teaching styles and student needs.

The screenshots for this essay are from the download version of GeoGebra 5. However, students using the online version at GeoGebra.org/classic will be able to follow the instructions with a little help from the instructor.

The lessons in this essay are a great way to introduce some Algebra I topics by having students investigate these topics visually. Many students need to see the Algebra rather than just memorize vocabulary and notes. Anecdotally, I have noticed a huge improvement in my students' cognitive understanding of these topics after using preliminary versions of these GeoGebra lessons in my own classes. Technology is an amazing advantage if teachers use it properly.

Lesson 1: Introduction to Slope-Intercept Form

Time for Lesson: 45 minutes

Summary: Students will use GeoGebra to investigate lines with positive slope. This lesson will focus on lines in slope-intercept form. The students will set up and manipulate their own GeoGebra worksheets to discover properties of lines with positive slope and answer guided questions. Students will be able to share their findings in class.

Materials:

- Guided student handout with investigative questions
- Devices with GeoGebra
- Teacher notes with answer key

Lesson Procedures:

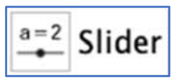
- Students will get their devices and the teacher will explain the lesson. (3 minutes)
- Students will complete the guided handout and interactive GeoGebra worksheet. (32 minutes)
  - The teacher will monitor students' progress and ask leading questions when appropriate.
  - As students complete each section of the guided handout, they will build their understanding by answering pointed questions.
  - Students will have to show the teacher their progress at certain checkpoints before they can move on, to ensure they are on the right track.
- The teacher will choose one or two students or student groups to present their GeoGebra worksheet and investigative questions. (10 minutes)
- The teacher will assign homework.

Introduction to Slope-Intercept Form (Student Worksheet)

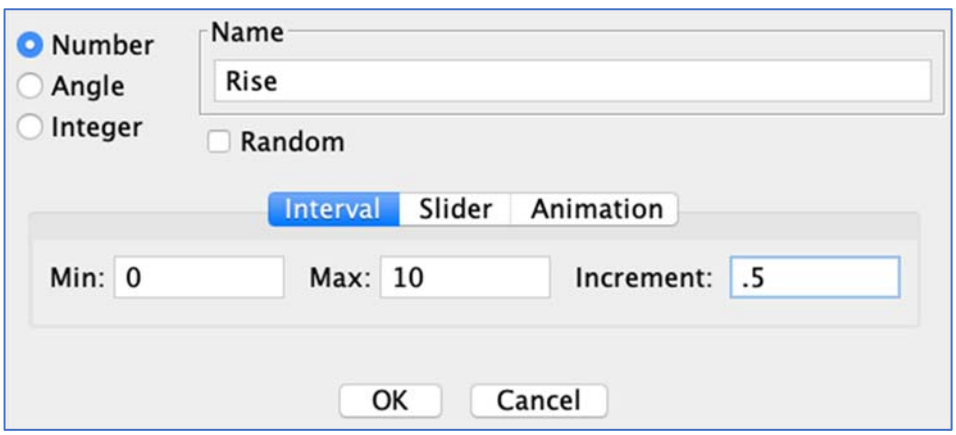
Directions: Follow the handout to create a GeoGebra worksheet about the slope-intercept form of a line. Answer the questions on the handout and be prepared to share your work with the class.

Section 1: Setting up your GeoGebra worksheet

1. Open GeoGebra on your device.
2. Using the 

tool, create an arbitrary point *A*.
3. Use the 

tool to create two variables, called "Rise" and "Run." For each slider, use 0 for the Min, 10 for the Max and 0.5 for the Increment, as in the image below.

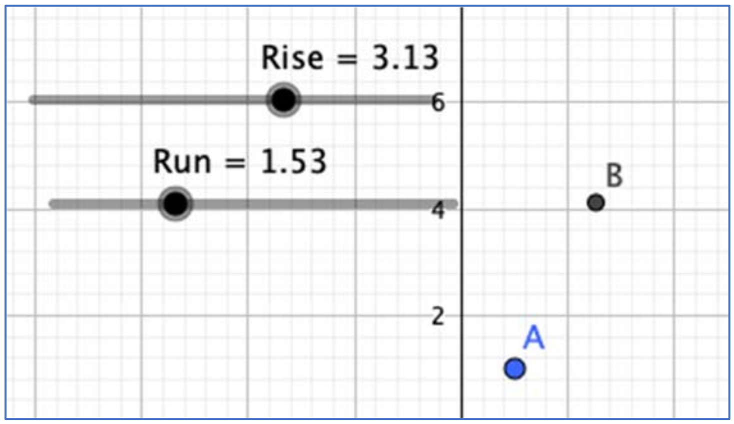


Note: Remember that variable names are case sensitive in GeoGebra. So GeoGebra treats "Rise" differently than "rise." Make sure that your equation for *B* in the next step uses the same capitalization that you used when creating your sliders.

4. For now, move the sliders so that neither “Rise” nor “Run” equals 0. Use “Rise” and “Run” to create a second point, *B*, by typing $B = A + (Run, Rise)$ into the 

bar. Then press enter.

Your GeoGebra worksheet should now look something like this:



Question 1:

a. As you move “Rise” and “Run” to different values, what do you notice about point *B* in relation to point *A*?

b. Is point *B* ever lower than point *A*? Why do you think this is the case?

c. Is point *B* ever further to the left than point *A*? Why do you think this is the case?

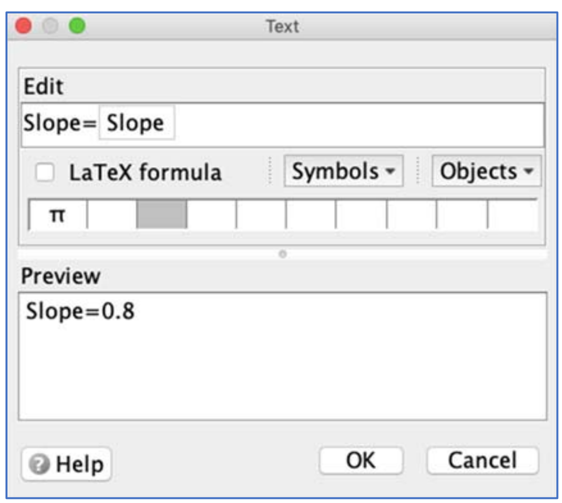
CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next section.

Section 2: Creating a line between two points

1. Use the Line tool to create a line through points *A* and *B*.
2. Right click on the line and choose the option “Equation: $y = mx + b$.” This form of the equation of a line is called *slope-intercept form* and we will be discovering more about this form later in the lesson.
3. The numerical equation of the line can be found in the Algebra pane in GeoGebra.
4. Next, type “$Slope = Rise / Run$” in the Input: bar.

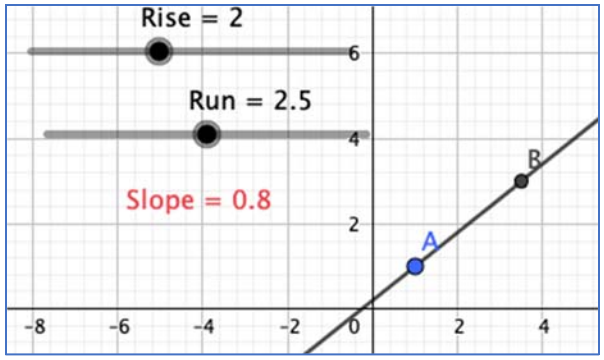
Definition: <u>Slope</u> is a measure of the steepness of a line, and is measured by the change in *y*-values divided by the change in *x*-values from one point to another on the line.

5. Using the ABC Text tool, create a text box so that you can easily see the value of the slope of the line. In the Edit region type “Slope =” and then use the Objects drop down menu to get a list of the objects that you have already created. Choose “Slope” to assign the created slope object to this text box. Then click “OK.” Refer to the image below.



Note: The Preview window in the Edit box above will show the text that will be added to the GeoGebra worksheet. The value of "Slope" will differ based on your worksheet.

Your GeoGebra worksheet should now look something like this:



Question 2: Use the sliders in the worksheet to change the values of "Rise" and "Run."

a. If we increase the value of "Rise" but do not increase the value of "Run," what happens to the value of "Slope?" What happens to the line?

b. If we increase the value of "Run" but do not increase the value of "Rise," what happens to the value of "Slope?" What happens to the line?

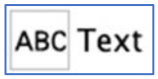
c. Earlier in this section, we saw that the general equation for the slope-intercept form of a line is $y = mx + b$. What connections do you notice between the value of "Slope" and the equation of the line in the Algebra window? Specifically, which letter does the value of "Slope" correspond to in the general slope-intercept equation?

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next section.

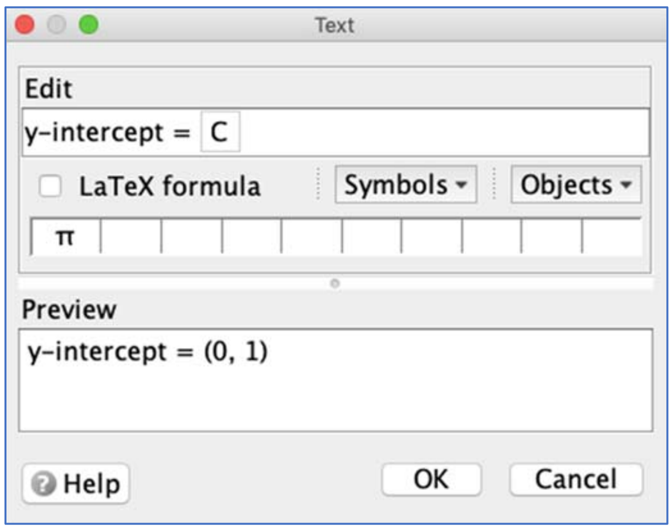
Section 3: Slope-intercept form and the *y*-intercept

1. Manipulate the sliders so that you can see where the line you created in Section 2 intersects the *y*-axis. Use the 

tool to construct the intersection point of the *y*-axis with the line. This will create point *C*.

2. Using the 

tool, create a text box to show the coordinates of point *C*. As you did when creating the text box for "Slope," use the Objects drop down menu to connect the text "*y*-intercept" with the object *C*, as shown below.



Definition: The <u>*y*-intercept</u> of a line is the point where the line crosses the *y*-axis.

Question 3: Use the sliders for "Rise" and "Run" to manipulate the line. Notice that point *C* moves to a new location.

a. Does the *y*-value of point *C* change?

b. Does the *x*-coordinate of point *C* change?

Question 4: Drag point *A* to a new position so that point *C* is also in a new location. In the general equation $y = mx + b$ for the line, which letter corresponds to point *C*?

CHECKPOINT: Be prepared to share your findings with the class. Work on the investigative questions below while other students are finishing their worksheets.

Further Investigation:

1. Set both "Rise" and "Run" to 0.
   a. What happens to the value of "Slope," the *y*-intercept, and the line?
   b. Why do you think this happens?
   c. What is the equation of the line in the Algebra pane when both "Rise" and "Run" are 0?
2. In your GeoGebra worksheet, is the value of "Slope" ever negative?
3. Is the *y*-coordinate for point *C* ever negative?

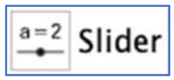
4. Set "Rise" to 0 and "Run" to a value greater than 0. What special kind of line does this create? What is the value of "Slope"? What do you notice about the *y*-coordinates of points *A*, *B*, and *C*?

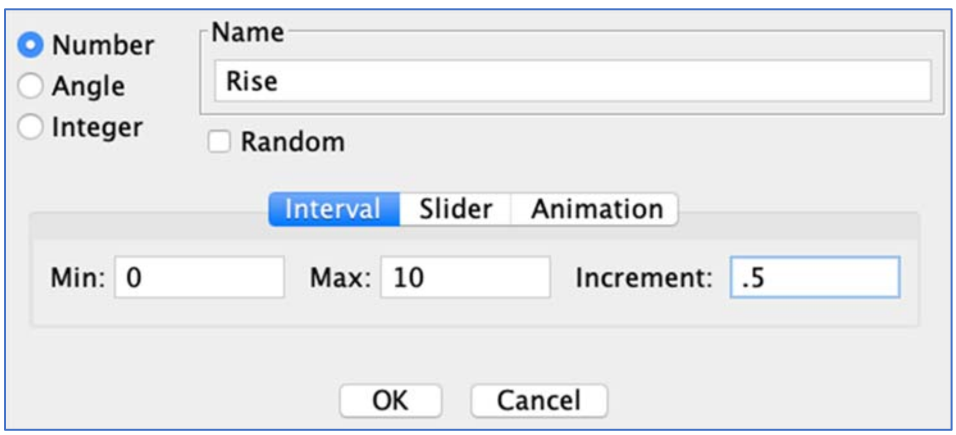
5. Set "Run" to 0 and "Rise" to a value greater than 0. What special kind of line does this create? What is the value of "Slope"? What do you notice about the *x*-coordinates of points *A* and *B*?

Introduction to Slope-Intercept Form (Teacher Version)

Directions: Follow the handout to create a GeoGebra worksheet about the slope-intercept form of a line. Answer the questions on the handout and be prepared to share your work with the class.

Section 1: Setting up your GeoGebra worksheet

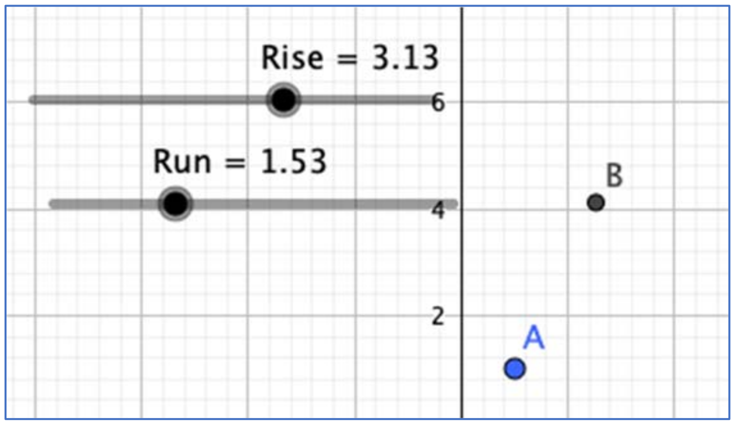
1. Open GeoGebra on your device.
2. Using the  tool, create an arbitrary point *A*.
3. Use the  tool to create two variables, called "Rise" and "Run." For each slider, use 0 for the Min, 10 for the Max and 0.5 for the Increment, as in the image below.



Note: Remember that variable names are case sensitive in GeoGebra. So GeoGebra treats "Rise" differently than "rise." Make sure that your equation for *B* in the next step uses the same capitalization that you used when creating your sliders.

4. For now, move the sliders so that neither "Rise" nor "Run" equals 0. Use "Rise" and "Run" to create a second point, *B*, by typing $B = A + (Run, Rise)$ into the  bar. Then press enter.

Your GeoGebra worksheet should now look something like this:



Question 1:

a. As you move "Rise" and "Run" to different values, what do you notice about point *B* in relation to point *A*?

As the value of "Rise" increases, the vertical distance from *A* to *B* increases. As the value of "Run" increases, the horizontal distance from *A* to *B* increases.

b. Is point *B* ever lower than point *A*? Why do you think this is the case?

Point *B* is never lower than point *A*, because we set 0 as the lowest value for the slider for "Rise."

c. Is point *B* ever further to the left than point *A*? Why do you think this is the case?

Point *B* is never further to the left than point *A*, because we set 0 as the lowest value for the slider for "Run."

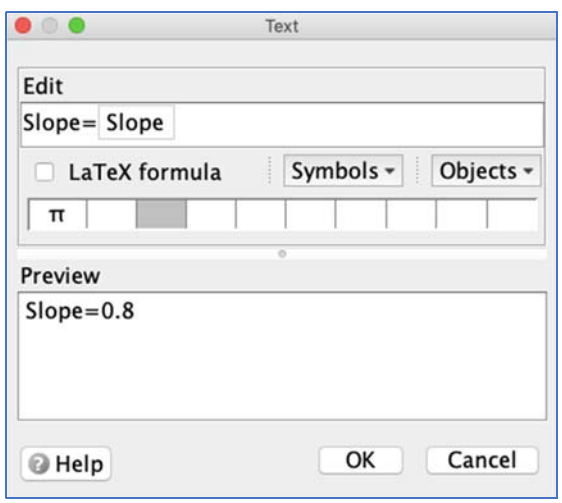
CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next section.

Section 2: Creating a line between two points

1. Use the **Line** tool to create a line through points *A* and *B*.
2. Right click on the line and choose the option "Equation: $y = mx + b$." This form of the equation of a line is called *slope-intercept form* and we will be discovering more about this form later in the lesson.
3. The numerical equation of the line can be found in the Algebra pane in GeoGebra.
4. Next, type "$Slope = Rise / Run$" in the **Input:** bar.

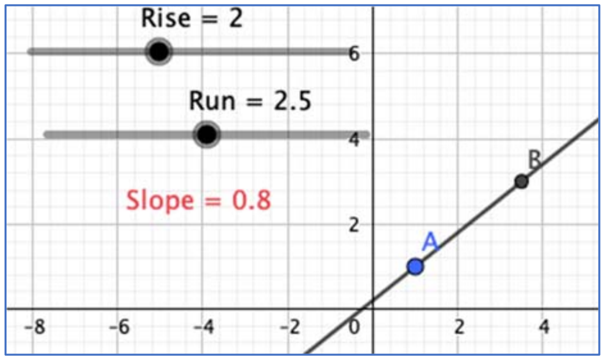
Definition: <u>Slope</u> is a measure of the steepness of a line, and is measured by the change in *y*-values divided by the change in *x*-values from one point to another on the line.

5. Using the **ABC Text** tool, create a text box so that you can easily see the value of the slope of the line. In the Edit region type "Slope =" and then use the **Objects** drop down menu to get a list of the objects that you have already created. Choose "Slope" to assign the created slope object to this text box. Then click "OK." Refer to the image below.



Note: The Preview window in the Edit box above will show the text that will be added to the GeoGebra worksheet. The value of “Slope” will differ based on your worksheet.

Your GeoGebra worksheet should now look something like this:



Question 2: Use the sliders in the worksheet to change the values of “Rise” and “Run.”

a. If we increase the value of “Rise” but do not increase the value of “Run,” what happens to the value of “Slope?” What happens to the line?

   The value of “Slope” becomes larger and the line becomes steeper.

b. If we increase the value of “Run” but do not increase the value of ”Rise,” what happens to the value of “Slope?” What happens to the line?

   The value of “Slope” becomes smaller and the line becomes less steep.

c. Earlier in this section, we saw that the general equation for the slope-intercept form of a line is $y = mx + b$. What connections do you notice between the value of “Slope” and the equation of the line in the Algebra window? Specifically, which letter does the value of “Slope” correspond to in the general slope-intercept equation?

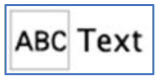
   The value of “Slope” is the number corresponding to *m* in the general slope-intercept equation.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next section.

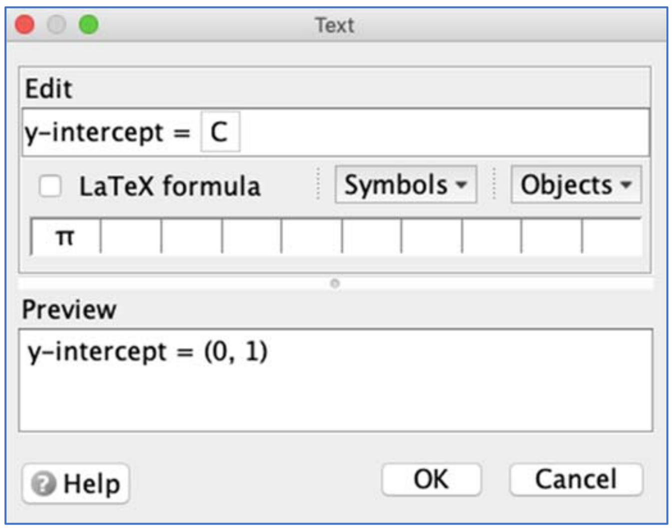
Section 3: Slope-intercept form and the *y*-intercept

1. Manipulate the sliders so that you can see where the line you created in Section 2 intersects the *y*-axis. Use the 

tool to construct the intersection point of the *y*-axis with the line. This will create point *C*.

2. Using the 

tool, create a text box to show the coordinates of point *C*. As you did when creating the text box for "Slope," use the Objects drop down menu to connect the text "*y*-intercept" with the object *C*, as shown below.



Definition: The <u>*y*-intercept</u> of a line is the point where the line crosses the *y*-axis.

Question 3: Use the sliders for "Rise" and "Run" to manipulate the line. Notice that point *C* moves to a new location.

a. Does the *y*-value of point *C* change?

The *y*-value does change as "Slope" changes because the point *C* moves as point *B* is moved.

b. Does the *x*-coordinate of point *C* change?

The *x*-value does not change because point *C* is always on the *y*-axis.

Question 4: Drag point *A* to a new position so that point *C* is also in a new location. In the general equation $y = mx + b$ for the line, which letter corresponds to point *C*?

The value of letter *b* changes as Point *C* changes. Students should note that *b* is the *y*-value of Point *C*.

CHECKPOINT: Be prepared to share your findings with the class. Work on the investigative questions below while other students are finishing their worksheets.

Further Investigation:

1. Set both "Rise" and "Run" to 0.
   a. What happens to the value of "Slope," the *y*-intercept, and the line?
      The equation in the text box for the line, the value of "Slope" and the *y*-intercept become "?" and point *B* is the same as point *A*. The line disappears.
   b. Why do you think this happens?
      If the value of "Rise" and "Run" are both 0 then the two points are at the same position and therefore there is no line. Without a line, there is no slope or *y*-intercept.
   c. What is the equation of the line in the Algebra pane when both "Rise" and "Run" are 0?
      GeoGebra shows that the line is "undefined." Make sure the students understand this is because there is no line when both "Rise" and "Run" are 0.
2. In your GeoGebra worksheet, is the value of "Slope" ever negative?
   The value of "Slope" will not be negative in this worksheet because neither "Run" nor "Rise" is negative. This goes back to the definition of slope.
3. Is the *y*-coordinate for point *C* ever negative?
   The *y*-coordinate of *C* becomes negative when point *C* goes below the *x*-axis.
   If the "Run" is set to zero and the students do not change it, point *C* will not be on the graph because there is no *y*-intercept.

4. Set "Rise" to 0 and "Run" to a value greater than 0. What special kind of line does this create? What is the value of "Slope"? What do you notice about the *y*-coordinates of points *A*, *B*, and *C*?

   The line is horizontal. The value of "Slope" is 0 and the *y*-intercept, point *A*, and point *B* have the same *y*-coordinate.

5. Set "Run" to 0 and "Rise" to a value greater than 0. What special kind of line does this create? What is the value of "Slope"? What do you notice about the *x*-coordinates of points *A* and *B*?

   The line is vertical. "Slope" is shown as "∞" in GeoGebra, and *A* and *B* have the same *x*-coordinate.

   During the class discussion, have a conversation with the students about the fact that the slope is actually undefined even though GeoGebra uses the symbol "∞." Be sure to explain that an undefined slope occurs because "Run" has a value of 0, which results in a quotient with a denominator of 0. Also point out that vertical lines do not have *y*-intercepts. This is because vertical lines, other than the *y*-axis itself, do not cross the *y*-axis.

Lesson 2: Positive and Negative Slope

Time for Lesson: 45 minutes

Summary: Students will use sliders in GeoGebra to investigate lines with positive and negative slope along with the *y*-intercept of these lines. Students will be able to share their findings in class.

Materials:

- Guided student handout with investigative questions
- Devices with GeoGebra
- Teacher notes with answer key

Lesson Procedures:

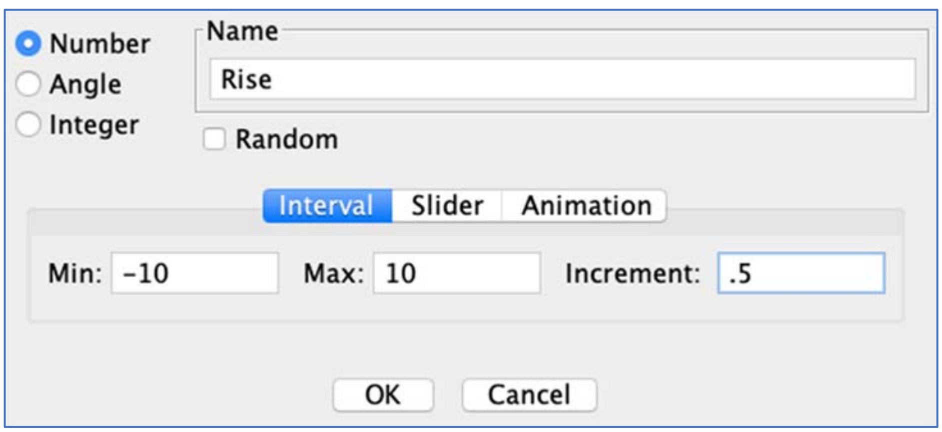
- Students will get their devices and the teacher will explain the lesson. (3 minutes)
- Students will complete the guided handout and interactive GeoGebra worksheet. (32 minutes)
  - The teacher will monitor students' progress and ask leading questions when appropriate.
  - As students complete each section of the guided handout, they will be asked pointed questions that help them develop a deeper understanding.
  - Students will have to show the teacher their progress at certain checkpoints before they can move on, to ensure that they are on the right track.
- The teacher will choose one or two students to present their GeoGebra worksheet and investigative questions. (10 minutes)
- The teacher will assign homework.

Positive and Negative Slope (Student Worksheet)

Directions: Follow the handout to create a GeoGebra worksheet about the slope-intercept form of a line. Answer the questions on the handout and be prepared to share your work with the whole class.

Section 1: Setting up the GeoGebra worksheet

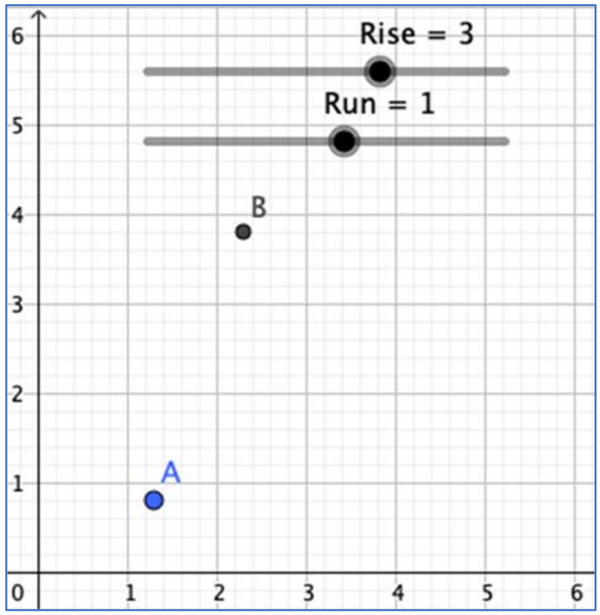
1. Open GeoGebra on your device.
2. Using the [A Point] tool, create an arbitrary point *A*.
3. Use the [a=2 Slider] tool to create two variables, called "Rise" and "Run." For each slider use -10 for the Min, 10 for the Max and 0.5 for the Increment, as in the image below.



Note: Remember that variable names are case sensitive in GeoGebra. So GeoGebra treats "Rise" differently than "rise." Make sure that your equation for *B* in the next step uses the same capitalization that you used when creating your sliders.

4. Move the sliders so that neither "Rise" nor "Run" is 0. Use "Rise" and "Run" to create point *B*, by typing $B = A + (Run, Rise)$ in the [Input:] bar.

Your GeoGebra worksheet should now look something like this:



Note: Take a few minutes to experiment with the sliders before answering the following questions.

Question 1:

a. Is point *B* ever lower than point *A*?

 - If so, is "Rise" positive or negative when *B* is lower than *A*?

 - If so, is "Run" positive or negative when *B* is lower than *A*?

b. Is point *B* ever further to the left than point *A*?

 - If so, is "Rise" positive or negative when *B* is further to the left than *A*?

 - If so, is "Run" positive or negative when *B* is further to the left than *A*?

c. For the following situations, make a conjecture as to whether "Rise" and "Run" will be either positive or negative.
   - Point B is lower and further to the right than point A.

   - Point B is lower and further to the left than point A.

   - Point B is higher and further to the right than point A.

   - Point B is higher and further to the left than point A.

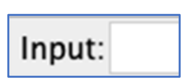
d. For each bullet in Question 1(c), set values of "Rise" and "Run" to match your answers.
   - Use the graph to check whether your conjectures are correct. If your conjectures are incorrect, use the GeoGebra worksheet and manipulate "Rise" and "Run" to investigate the inconsistencies and develop new conjectures.

   - Were there any exceptions?

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next section.

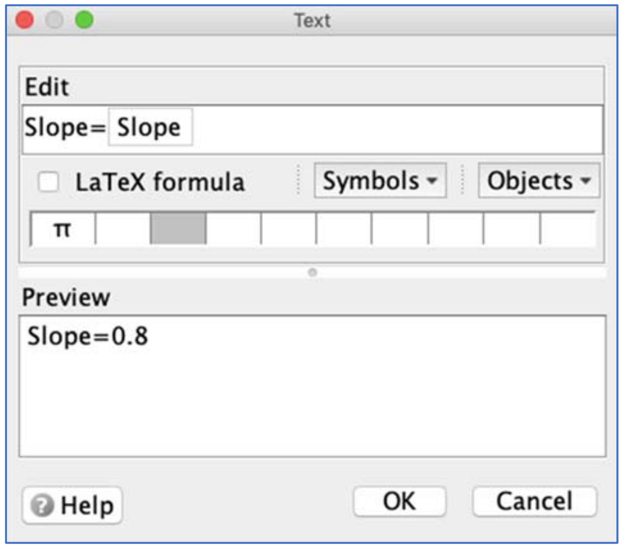
Section 2: Creating a line between the two points and investigating the slope of the line

1. Create a line through points A and B using the 

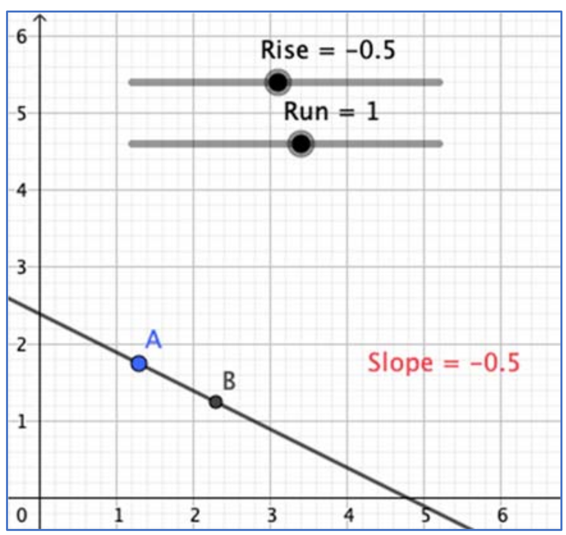
tool.
2. Right click on the line and choose the option "Equation: $y = mx + b$."
3. Type "$Slope = Rise / Run$" in the 

bar to compute the slope of the line.

4. Next, use the **ABC Text** tool to create a text box that displays the value of "Slope." Remember to attach the object "Slope" to the word "Slope" in the text box. Refer to the image below.



Your GeoGebra worksheet should now look something like this:



Question 2:

a. Set "Rise" to a value greater than zero and "Run" to a value greater than zero. What do you notice about "Slope" and the location of point *B* relative to point *A*?

b. Set "Rise" to a value greater than zero and "Run" to a value less than zero. What do you notice about "Slope" and the location of point *B* relative to point *A*?

c. Likewise, set "Rise" to a value less than zero and "Run" to a value greater than zero. What do you notice about "Slope" and the location of point *B* relative to point *A*?

d. Now, set the values of "Rise" and "Run" so they are both less than zero. What do you notice about "Slope" and the location of point *B* relative to point *A*?

e. Using your knowledge of slope and the values of "Rise" and "Run," explain why "Slope" has the characteristics that you described in parts (a), (b), (c) and (d). Be ready to share these findings with the class.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next section!

Section 3: Finding the *y*-intercept and further investigation

1. Remember that the <u>*y*-intercept</u> of the line is the intersection of the line with the *y*-axis. Move the sliders for "Rise" and "Run" until the *y*-intercept of the line is in the viewing window.
2. Using the 

tool, plot the *y*-intercept of the line. This point should be labeled *C*.
3. Create a text box for the *y*-intercept connected to the new point *C*, so you can easily see the value of the *y*-intercept while manipulating the line.

Question 3:

a. Manipulate the sliders so that "Slope" is negative. Can the *y*-intercept be:
   i. Above the *x*-axis?

   ii. Below the *x*-axis?

b. Manipulate the sliders so that "Slope" is positive. Can the *y*-intercept be:
   i. Above the *x*-axis?

   ii. Below the *x*-axis?

c. Based on your investigation, is there a connection between the value of "Slope" and the location of the *y*-intercept of the line?

Question 4:

a. Describe the difference in appearance between a line with positive value of "Slope" and a line with negative value of "Slope".

b. Consider the following scenarios. In each situation, would the slope of the line be positive or negative?
   - Positive "Rise" and positive "Run:"
   - Positive "Rise" and negative "Run:"
   - Negative "Rise" and positive "Run:"
   - Negative "Rise" and negative "Run:"

c. Experiment with the sliders. Can you ever get a line with no *y*-intercept?
   - If so, what value of "Slope" does such a line have?

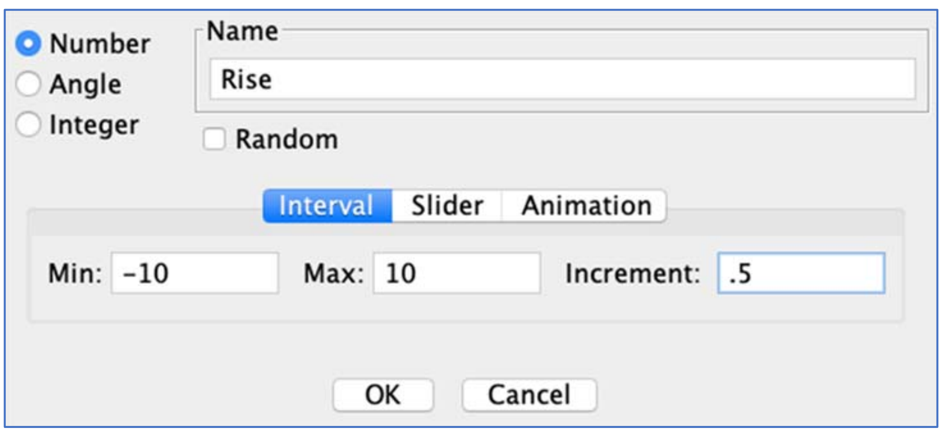
CHECKPOINT: Be prepared to share your findings with the class.

Positive and Negative Slope (Teacher Version)

Directions: Follow the handout to create a GeoGebra worksheet about the slope-intercept form of a line. Answer the questions on the handout and be prepared to share your work with the whole class.

Section 1: Setting up the GeoGebra worksheet

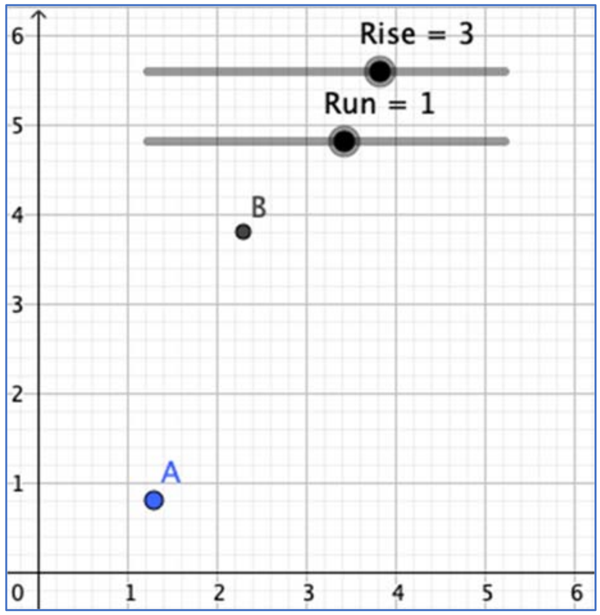
1. Open GeoGebra on your device.
2. Using the Point tool, create an arbitrary point *A*.
3. Use the Slider tool to create two variables, called "Rise" and "Run." For each slider use -10 for the Min, 10 for the Max and 0.5 for the Increment, as in the image below.



Note: Remember that variable names are case sensitive in GeoGebra. So GeoGebra treats "Rise" differently than "rise." Make sure that your equation for *B* in the next step uses the same capitalization that you used when creating your sliders.

4. Move the sliders so that neither "Rise" nor "Run" is 0. Use "Rise" and "Run" to create point *B*, by typing $B = A + (Run, Rise)$ in the Input: bar.

Your GeoGebra worksheet should now look something like this:



Note: Take a few minutes to experiment with the sliders before answering the following questions.

Question 1:

a. Is point *B* ever lower than point *A*?

   Point *B* can be lower than point *A.*

   - If so, is "Rise" positive or negative when *B* is lower than *A*?

     "Rise" is negative when *B* is lower than *A.*

   - If so, is "Run" positive or negative when *B* is lower than *A*?

     "Run" can be positive, negative, or zero.

b. Is point *B* ever further to the left than point *A*?

   Point *B* can be further to the left than point *A.*

   - If so, is "Rise" positive or negative when *B* is further to the left than *A*?

     "Rise" can be positive, negative, or zero.

   - If so, is "Run" positive or negative when *B* is further to the left than *A*?

     "Run" is negative when *B* is further to the left than *A.*

c. For the following situations, make a conjecture as to whether "Rise" and "Run" will be either positive or negative.

- Point *B* is lower and further to the right than point *A*.

  "Rise" is negative and "Run" is positive.

- Point *B* is lower and further to the left than point *A*.

  "Rise" is negative and "Run" is negative.

- Point *B* is higher and further to the right than point *A*.

  "Rise" is positive and "Run" is positive.

- Point *B* is higher and further to the left than point *A*.

  "Rise" is positive and "Run" is negative.

d. For each bullet in Question 1(c), set values of "Rise" and "Run" to match your answers.

- Use the graph to check whether your conjectures are correct. If your conjectures are incorrect, use the GeoGebra worksheet and manipulate "Rise" and "Run" to investigate the inconsistencies and develop new conjectures.

  Make sure the students' conjectures match the answers in (c).

- Were there any exceptions?

  There should be no exceptions.

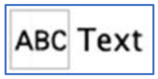
CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next section.

Section 2: Creating a line between the two points and investigating the slope of the line

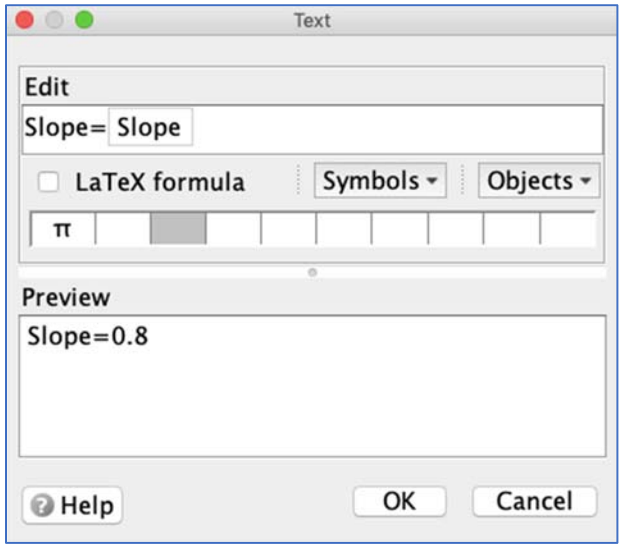
1. Create a line through points *A* and *B* using the 

tool.
2. Right click on the line and choose the option "Equation: $y = mx + b$."
3. Type "$Slope = Rise / Run$" in the 

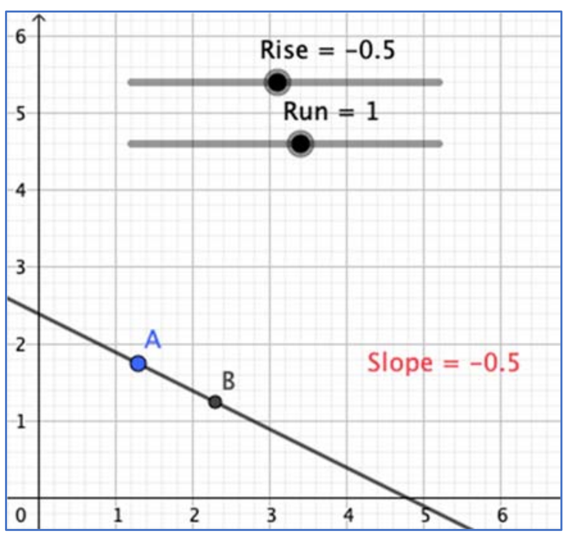
bar to compute the slope of the line.

4. Next, use the 

tool to create a text box that displays the value of "Slope." Remember to attach the object "Slope" to the word "Slope" in the text box. Refer to the image below.



Your GeoGebra worksheet should now look something like this:



Question 2:

a. Set "Rise" to a value greater than zero and "Run" to a value greater than zero. What do you notice about "Slope" and the location of point *B* relative to point *A*?

Point *B* is above and to the right of Point *A* and "Slope" is positive.

b. Set "Rise" to a value greater than zero and "Run" to a value less than zero. What do you notice about "Slope" and the location of point *B* relative to point *A*?

Point *B* is above and to the left of point *A* and "Slope" is negative.

c. Likewise, set "Rise" to a value less than zero and "Run" to a value greater than zero. What do you notice about "Slope" and the location of point *B* relative to point *A*?

Point *B* is below and to the right of point *A* and "Slope" is still negative.

d. Now, set the values of "Rise" and "Run" so they are both less than zero. What do you notice about "Slope" and the location of point *B* relative to point *A*?

Point *B* is below and to the left of point *A* and "Slope" is positive.

e. Using your knowledge of slope and the values of "Rise" and "Run," explain why "Slope" has the characteristics that you described in parts (a), (b), (c) and (d). Be ready to share these findings with the class.

Since "Slope" is a quotient of "Rise" over "Run", when either "Rise" or "Run" is negative and the other is positive, then "Slope" will be negative. If both "Rise" and "Run" are positive or both "Rise" and "Run" are negative, then "Slope" will be positive.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next section!

Section 3: Finding the *y*-intercept and further investigation

1. Remember that the <u>*y*-intercept</u> of the line is the intersection of the line with the *y*-axis. Move the sliders for "Rise" and "Run" until the *y*-intercept of the line is in the viewing window.
2. Using the 

tool, plot the *y*-intercept of the line. This point should be labeled *C*.
3. Create a text box for the *y*-intercept connected to the new point *C*, so you can easily see the value of the *y*-intercept while manipulating the line.

Question 3:

a. Manipulate the sliders so that "Slope" is negative. Can the *y*-intercept be:
   i. Above the *x*-axis?
      The *y*-intercept can be above the *x*-axis when "Slope" is negative.
   ii. Below the *x*-axis?
      The *y*-intercept can be below the *x*-axis when "Slope" is negative.

b. Manipulate the sliders so that "Slope" is positive. Can the *y*-intercept be:
   i. Above the *x*-axis?
      The *y*-intercept can be above the *x*-axis when "Slope" is positive.
   ii. Below the *x*-axis?
      The *y*-intercept can be below the *x*-axis when "Slope" is positive.

c. Based on your investigation, is there a connection between the value of "Slope" and the location of the *y*-intercept of the line?
   The value of the "Slope" and the value of the *y*-intercept are not connected in that the value of "Slope" does not imply a location for the *y*-intercept relative to the *x*-axis.

Question 4:

a. Describe the difference in appearance between a line with positive value of "Slope" and a line with negative value of "Slope".

   If the value of "Slope" is positive, the line will go "upwards" from left to right in the coordinate plane. That is, the line is increasing.

   Likewise, if the value of "Slope" is negative, the line will go "downwards" from left to right. That is, the line is decreasing.

b. Consider the following scenarios. In each situation, would the slope of the line be positive or negative?

   - Positive "Rise" and positive "Run:" Positive "Slope"
   - Positive "Rise" and negative "Run:" Negative "Slope"
   - Negative "Rise" and positive "Run:" Negative "Slope"
   - Negative "Rise" and negative "Run:" Positive "Slope"

c. Experiment with the sliders. Can you ever get a line with no *y*-intercept?

   Explain to the students that not being able to see the intercept in the GeoGebra window does not mean that the *y*-intercept does not exist. Sometimes the intercept is simply outside the viewing window of the graph.

   Students may report that there is no *y*-intercept when the *y*-intercept is "?." Lines that do not have a *y*-intercept are vertical lines with a "Slope" that is undefined. In this case, GeoGebra indicates that the intercept is "?" which means it does not exist.

   - If so, what value of "Slope" does such a line have?

     GeoGebra indicates that for vertical lines "Slope" is "∞." During the class discussion remind students that the slope of the line is actually undefined and not "∞" as GeoGebra indicates. Be sure to explain that an undefined slope occurs when "Run" has a value of 0. This results in "Slope" being a quotient with a denominator of 0.

CHECKPOINT: Be prepared to share your findings with the class.

Lesson 3: Computing Slope by Finding Rise and Run Between Two Points

Time for Lesson: 45 minutes

Summary: In the last two lessons, students used a point and values of "Rise" and "Run" to create a second point. Then they used these points to form a line. Now, students will use two points on an existing line to compute "Rise" and "Run." They will then investigate positive and negative slope.

Materials:

- Guided student handout with investigative questions
- Devices with GeoGebra
- Teacher notes with answer key

Lesson Procedures:

- Students will access their devices and the teacher will explain the lesson. (3 minutes)
- Students will complete the guided handout and interactive GeoGebra worksheet. (32 minutes)
  - The teacher will monitor students' progress and ask leading questions when appropriate.
  - As students complete each section of the guided handout, they will be asked pointed questions that develop a deeper understanding.
  - Students will have to show the teacher their progress at certain checkpoints before they can move on, to ensure they are on the right track.
- The teacher will choose one or two students to present their GeoGebra worksheet and investigative questions. (10 minutes)
- The teacher will assign homework.

Computing Slope by Finding Rise and Run Between Two Points (Student Worksheet)

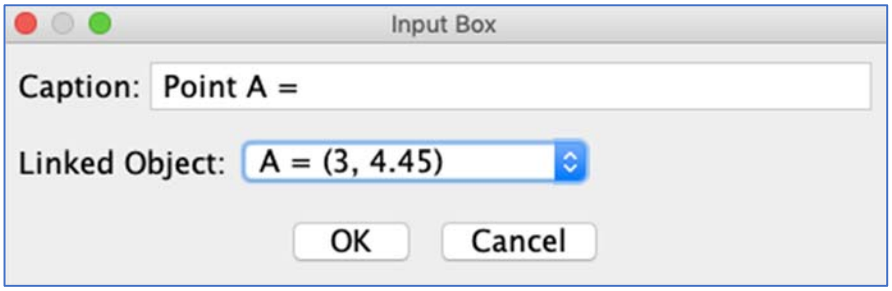
Directions: Follow the handout to create a GeoGebra worksheet about computing slope by finding "Rise" and "Run" between two points. Answer the questions on the handout and be prepared to share your work with the whole class. Capitalize the variables exactly as shown in the handout.

Section 1: Setting up the GeoGebra worksheet

1. Open GeoGebra on your device.
2. Using the 

tool, create arbitrary points *A* and *B* in GeoGebra.
3. Using the 

tool, create an input box captioned "Point *A* =." Make sure the linked object is point *A*, as in the image below, and then click "OK."



4. Create an input box for point *B* in the same way.

Question 1:

a. The coordinates of a point on a graph are given in the form (x, y). What do x and y represent on the graph?

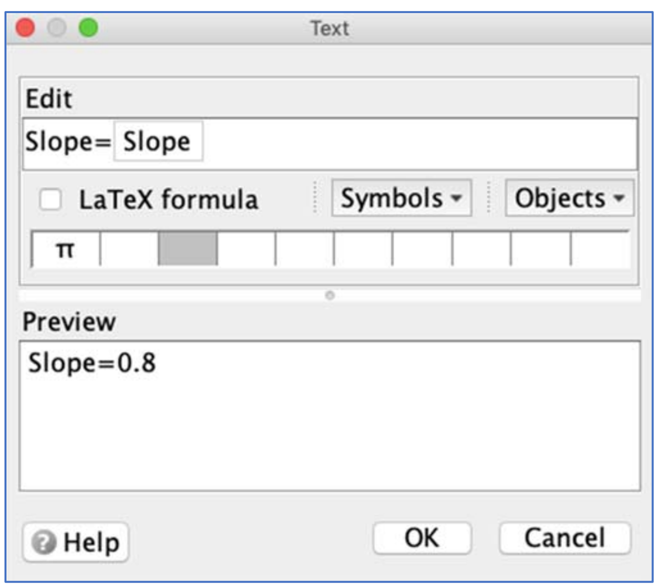
b. Which coordinate of a point changes when the point is moved vertically?

c. Which coordinate of a point changes when the point is moved horizontally?

Section 2: Creating slope of a line between two points

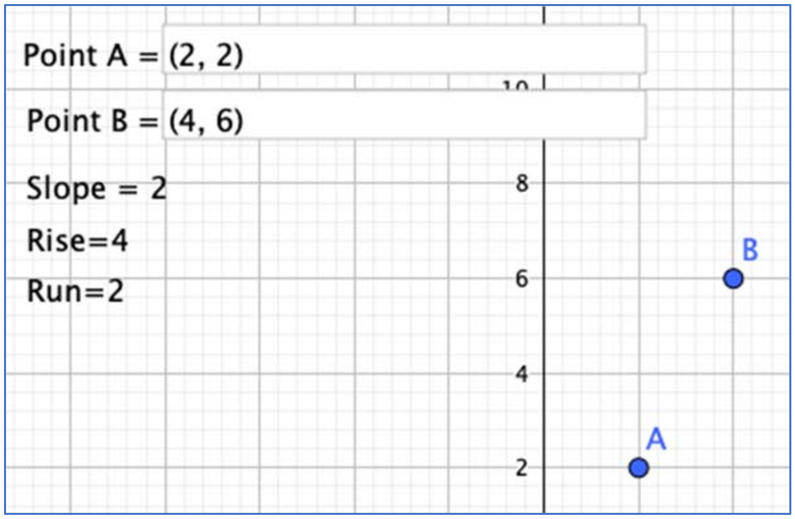
1. In the **Input:** bar, type "$Rise = \left(y(B) - y(A)\right)$" and click enter.
   - The "Rise" is the change in *y*-values from one point to another. From *A* to *B*, the "Rise" is the *y*-value of *B* minus the *y*-value of *A*.
2. Similarly, in the input bar, type "$Run = \left(x(B) - x(A)\right)$" and click enter.
   - The "Run" is the change in *x*-values from one point to another. From *A* to *B* the "Run" is the *x*-value of *B* minus the *x*-value of *A*.
3. Use "Rise" and "Run" in the GeoGebra worksheet to compute the slope of the line between *A* and *B*: In the input bar, type "$Slope = Rise / Run$."
4. Next, use the **ABC Text** tool to create a text box that displays the value of "Slope."

   Remember to attach the object "Slope" in your text box to the word "Slope." Refer to the image below.



5. Create a text box for “Rise” and “Run” in the same way as above.

Your GeoGebra worksheet should now look something like this:



Question 2:

a. Set the *y*-coordinates of points *A* and *B* to the same value. Make sure the *x*-coordinates are different. What do you notice about the position of point *A* relative to *B*?

- What is the value of “Slope?”

- Change the *y*-coordinates of *A* and *B* so that the two points still have the same *y*-coordinate as each other. What is the value of “Slope?”

- Without drawing the line, describe the type of line that passes through *A* and *B.*

b. Change the coordinates of *A* and *B* so that the *x*-coordinates of both points are the same and the *y*-coordinates are different. What do you notice about the position of point *A* relative to point *B?*

- What does GeoGebra indicate as the "Slope" of the line?

- What is the actual slope of the line?

- Change the *x*-coordinates of *A* and *B* so that the two points still have the same *x*-coordinate as each other. What is the value of "Slope?"

- Without drawing the line, describe the type of line that passes through *A* and *B.*

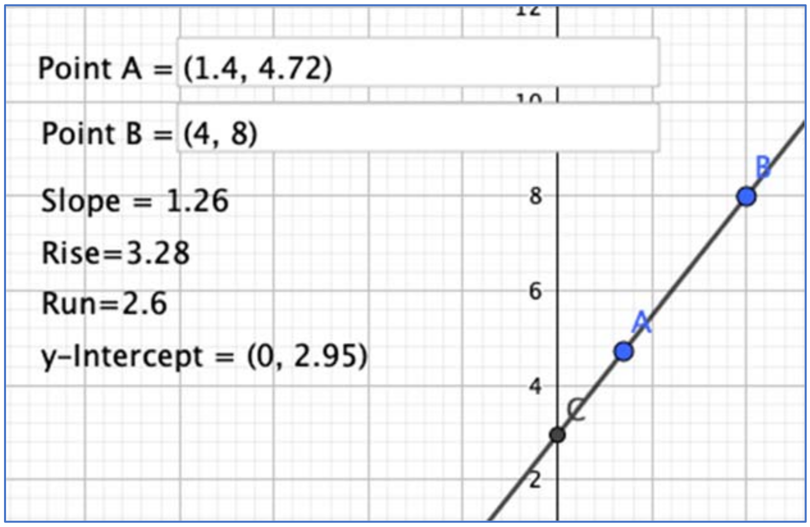
Section 3: Creating the line and *y*-intercept

1. Use the 

tool to create a line through point *A* and point *B.*
2. Drag Point *A* so the line is not vertical.
3. Right click on the line and choose the option "Equation: $y = mx + b$."
4. Move the sliders for "Rise" and "Run" until the *y*-intercept of the line is in the viewing window.
5. Using the 

tool, plot the *y*-intercept of the line as the intersection between the line and the *y*-axis. This point should be labeled *C.*
6. Finally, create a text box for the *y*-intercept, connected to point *C*, so you can easily see its coordinates.

Your GeoGebra worksheet should now look something like this:



Question 3:

a. Change the coordinates of points *A* and *B* if necessary so that *A* and *B* are different points. Show how to compute "Rise" using the specific values of the coordinates of *A* and *B*.

b. Show how to compute "Run" using the specific values of the coordinates of *A* and *B*.

Question 4:

a. Set the coordinates of *A* and *B* so the *x*-coordinate of *B* is greater than the *x*-coordinate of *A* and the *y*-coordinate of *B* is greater than the *y*-coordinate of *A*.
   - Does the line have a positive or negative value of "Slope"?

   - Why do you think this is true?

- Change the coordinates of *A* and *B* so that both coordinates of *B* are still greater than the corresponding coordinates of *A*. Does the sign of "Slope" change?

b. Set the coordinates of *A* and *B* so the *x*-coordinate of *B* is less than the *x*-coordinate of *A* and the *y*-coordinate of *B* is greater than the *y*-coordinate of *A*.
   - Does the line have a positive or negative value of value of "Slope?"

   - Why do you think this is true?

   - Change the coordinates of *A* and *B* so the *x*-coordinate of *B* is still less than the *x*-coordinate of *A* and the *y*-coordinate of *B* is still greater than the *y*-coordinate of *A*. Does the sign of "Slope" change?

c. Set the coordinates of *A* and *B* so the *x*-coordinate of *B* is greater than the *x*-coordinate of *A* and the *y*-coordinate of *B* is less than the *y*-coordinate of *A*.
   - Does the line have a positive or negative value of "Slope"?

   - Why do you think this is true?

- Change the coordinates of *A* and *B* so the *x*-coordinate of *B* is still greater than the *x*-coordinate of *A* and the *y*-coordinate of *B* is still less than the *y*-coordinate of *A*. Does the sign of “Slope” change?

d. Set the coordinates of *A* and *B* so the *x*-coordinate of *B* is less than the *x*-coordinate of *A* and the *y*-coordinate of *B* is less than the *y*-coordinate of *A*.

- Does the line have a positive or negative “Slope”?

- Why do you think this is true?

- Change the coordinates of *A* and *B* so the *x*-coordinate of *B* is still less than the *x*-coordinate of *A* and the *y*-coordinate of *B* is still less than the *y*-coordinate of *A*. Does the sign of “Slope” change?

Question 5:

a. Set the *y*-coordinates of *A* and *B* to the same value and the *x*-coordinates to different values, as in Question 2(a).

- How is the *y*-intercept related to the *y*-coordinates of *A* and *B?*

- Change the *y*-coordinates of *A* and *B* so that the two points still have the same *y*-coordinate as each other. Does the relationship between the *y*-intercept and the *y*-coordinates of *A* and *B* remain the same?

b. Set the *x*-coordinates of *A* and *B* to the same value and the *y*-coordinates to different values, as in Question 2(b).

- What does GeoGebra indicate as the *y*-intercept of the line?

- Why do you think this is true?

- Change the *x*-coordinates of *A* and *B* so that the two points still have the same *x*-coordinate as each other. Does the line now have a *y*-intercept?

Question 6:

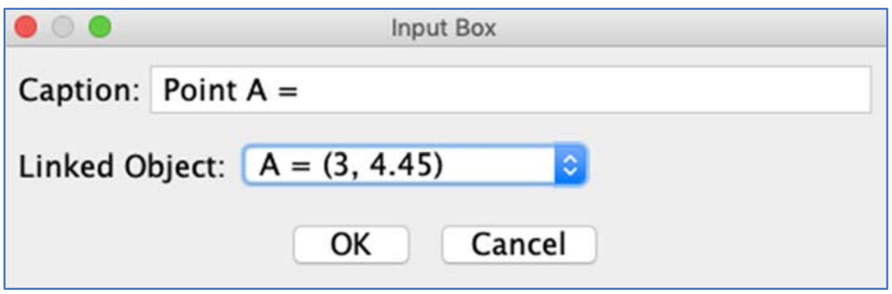
Formulate a brief overview of the slope and *y*-intercept of a line. Write at least three facts about slope and *y*-intercept and be able to explain them to the class. Keep in mind that you should be able to explain as if your classmates have never learned about slope and *y*-intercept before. Your explanation may use what you have observed in all of the previous lessons.

Computing Slope by Finding Rise and Run Between Two Points (Teacher Version)

Directions: Follow the handout to create a GeoGebra worksheet about computing slope by finding "Rise" and "Run" between two points. Answer the questions on the handout and be prepared to share your work with the whole class. Capitalize the variables exactly as shown in the handout.

Section 1: Setting up the GeoGebra worksheet

1. Open GeoGebra on your device.
2. Using the [A Point] tool, create arbitrary points *A* and *B* in GeoGebra.
3. Using the [a=1 Input Box] tool, create an input box captioned "Point *A* =." Make sure the linked object is point *A*, as in the image below, and then click "OK."



4. Create an input box for point *B* in the same way.

Question 1:

a. The coordinates of a point on a graph are given in the form (x, y). What do x and y represent on the graph?

The first coordinate (x) represents the point's horizontal location relative to the origin.

The second coordinate (y) represents the point's vertical location relative to the origin.

b. Which coordinate of a point changes when the point is moved vertically?

The *y*-coordinate (second coordinate)

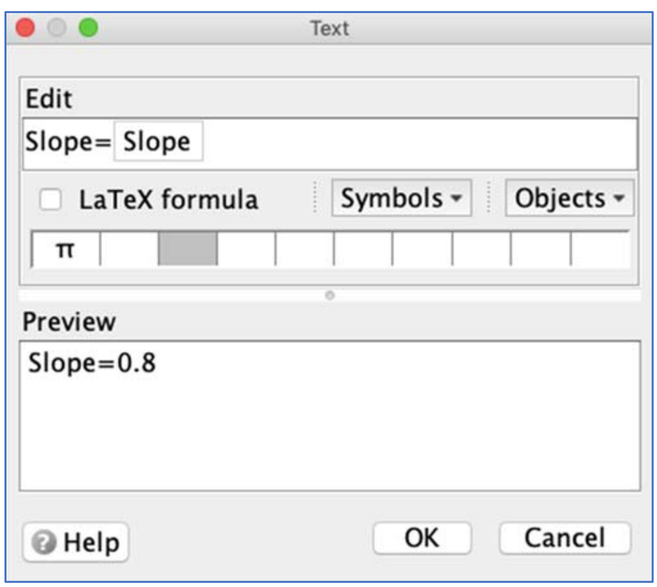
c. Which coordinate of a point changes when the point is moved horizontally?

The *x*-coordinate (first coordinate)

Section 2: Creating slope of a line between two points

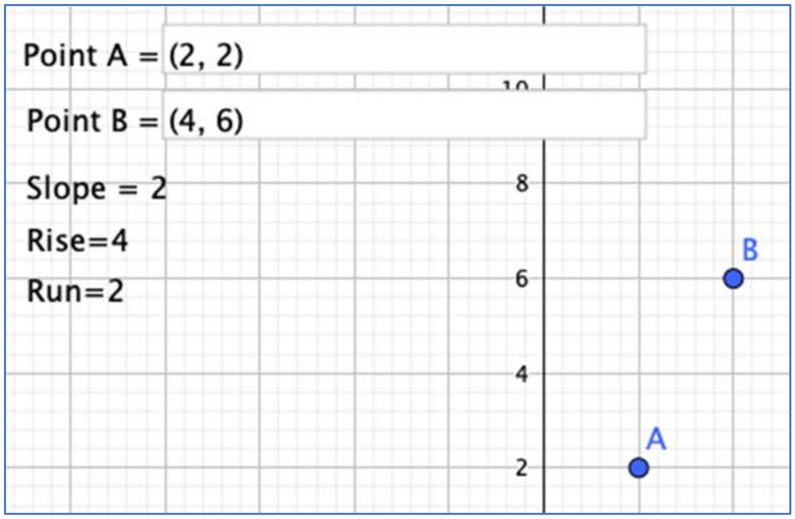
1. In the Input: bar, type "$Rise = (y(B) - y(A))$" and click enter.
   - The "Rise" is the change in *y*-values from one point to another. From *A* to *B*, the "Rise" is the *y*-value of *B* minus the *y*-value of *A*.
2. Similarly, in the input bar, type "$Run = (x(B) - x(A))$" and click enter.
   - The "Run" is the change in *x*-values from one point to another. From *A* to *B* the "Run" is the *x*-value of *B* minus the *x*-value of *A*.
3. Use "Rise" and "Run" in the GeoGebra worksheet to compute the slope of the line between *A* and *B*: In the input bar, type "$Slope = Rise / Run$."
4. Next, use the ABC Text tool to create a text box that displays the value of "Slope."

   Remember to attach the object "Slope" in your text box to the word "Slope." Refer to the image below.



5. Create a text box for “Rise” and “Run” in the same way as above.

Your GeoGebra worksheet should now look something like this:



Question 2:

a. Set the *y*-coordinates of points *A* and *B* to the same value. Make sure the *x*-coordinates are different. What do you notice about the position of point *A* relative to *B*?

Point *A* is directly to the left or directly to the right of point *B*.

- What is the value of “Slope?”

  The value of “Slope” is 0.

- Change the *y*-coordinates of *A* and *B* so that the two points still have the same *y*-coordinate as each other. What is the value of “Slope?”

  The value of “Slope” is still 0.

- Without drawing the line, describe the type of line that passes through *A* and *B*.

  A horizontal line

b. Change the coordinates of *A* and *B* so that the *x*-coordinates of both points are the same and the *y*-coordinates are different. What do you notice about the position of point *A* relative to point *B?*

Point *A* is directly above or directly below point *B.*

- What does GeoGebra indicate as the "Slope" of the line?

  GeoGebra indicates that "Slope" is ∞.

- What is the actual slope of the line?

  The slope is undefined. During the class discussion remind students that the slope is actually undefined and not "∞" as GeoGebra indicates. Be sure to explain that an undefined slope occurs when "Run" has a value of 0. This results in "Slope" being a quotient with a denominator of 0.

- Change the *x*-coordinates of *A* and *B* so that the two points still have the same *x*-coordinate as each other. What is the value of "Slope?"

  The value of "Slope" is still undefined.

- Without drawing the line, describe the type of line that passes through *A* and *B.*

  A vertical line

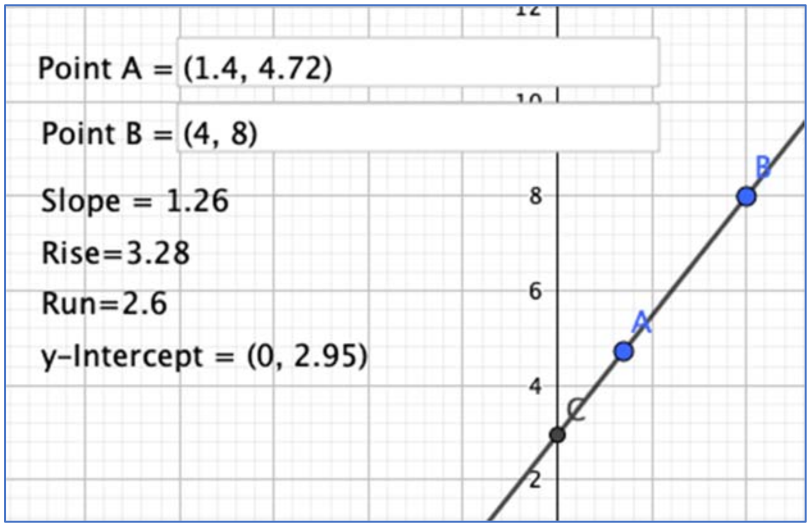
Section 3: Creating the line and *y*-intercept

1. Use the 

tool to create a line through point *A* and point *B.*
2. Drag Point *A* so the line is not vertical.
3. Right click on the line and choose the option "Equation: $y = mx + b$."
4. Move the sliders for "Rise" and "Run" until the *y*-intercept of the line is in the viewing window.
5. Using the 

tool, plot the *y*-intercept of the line as the intersection between the line and the *y*-axis. This point should be labeled *C*.
6. Finally, create a text box for the *y*-intercept, connected to point *C*, so you can easily see its coordinates.

Your GeoGebra worksheet should now look something like this:



Question 3:

a. Change the coordinates of points *A* and *B* if necessary so that *A* and *B* are different points. Show how to compute "Rise" using the specific values of the coordinates of *A* and *B*.

Answers vary, but should be an arithmetic expression such as $Rise = 3 - 2$.

b. Show how to compute "Run" using the specific values of the coordinates of *A* and *B*.

Answers vary, but should be an arithmetic expression such as $Run = 5 - 3$.

Question 4:

a. Set the coordinates of *A* and *B* so the *x*-coordinate of *B* is greater than the *x*-coordinate of *A* and the *y*-coordinate of *B* is greater than the *y*-coordinate of *A*.

- Does the line have a positive or negative value of "Slope"?

  positive

- Why do you think this is true?

  Since both the *x*-coordinate and *y*-coordinate are greater for *B* than for *A*, the values of "Run" and "Rise" are both positive. Since "Slope" is a quotient of a positive value with a positive value, then "Slope" has a positive value.

- Change the coordinates of *A* and *B* so that both coordinates of *B* are still greater than the corresponding coordinates of *A*. Does the sign of "Slope" change?

  No, the sign of "Slope" remains the same.

b. Set the coordinates of *A* and *B* so the *x*-coordinate of *B* is less than the *x*-coordinate of *A* and the *y*-coordinate of *B* is greater than the *y*-coordinate of *A*.

- Does the line have a positive or negative value of value of "Slope?"

  negative

- Why do you think this is true?

  Since the *x*-coordinate of *B* is less than the *x*-coordinate of *A*, "Run" is negative. Likewise, since the *y*-coordinate of *B* is greater than the *y*-coordinate of *A*, "Rise" is positive. Since "Slope" is a quotient of a positive value with a negative value, then "Slope" has a negative value.

- Change the coordinates of *A* and *B* so the *x*-coordinate of *B* is still less than the *x*-coordinate of *A* and the *y*-coordinate of *B* is still greater than the *y*-coordinate of *A*. Does the sign of "Slope" change?

  No, the sign of "Slope" remains the same.

c. Set the coordinates of *A* and *B* so the *x*-coordinate of *B* is greater than the *x*-coordinate of *A* and the *y*-coordinate of *B* is less than the *y*-coordinate of *A*.

- Does the line have a positive or negative value of "Slope"?

  negative

- Why do you think this is true?

  Since the *x*-coordinate of *B* is greater than the *x*-coordinate of *A*, "Run" is positive. Likewise, since the *y*-coordinate of *B* is less than the *y*-coordinate of *A*, "Rise" is negative. Since "Slope" is a quotient of a negative value with a positive value, then "Slope" has a negative value.

- Change the coordinates of *A* and *B* so the *x*-coordinate of *B* is still greater than the *x*-coordinate of *A* and the *y*-coordinate of *B* is still less than the *y*-coordinate of *A*. Does the sign of "Slope" change?

  No, the sign of "Slope" remains the same.

d. Set the coordinates of *A* and *B* so the *x*-coordinate of *B* is less than the *x*-coordinate of *A* and the *y*-coordinate of *B* is less than the *y*-coordinate of *A*.

- Does the line have a positive or negative "Slope"?

  positive

- Why do you think this is true?

  Since the *x*-coordinate and *y*-coordinate values are greater for *A* than for *B*, both "Rise" and "Run" are negative. Since "Slope" is a quotient of a negative value with a negative value, then "Slope" has a positive value.

- Change the coordinates of *A* and *B* so the *x*-coordinate of *B* is still less than the *x*-coordinate of *A* and the *y*-coordinate of *B* is still less than the *y*-coordinate of *A*. Does the sign of "Slope" change?

  No, the sign of "Slope" remains the same.

Question 5:

a. Set the *y*-coordinates of *A* and *B* to the same value and the *x*-coordinates to different values, as in Question 2(a).

- How is the *y*-intercept related to the *y*-coordinates of *A* and *B?*

  The *y*-coordinate of *C* is the same as that for *A* and *B.*

- Change the *y*-coordinates of *A* and *B* so that the two points still have the same *y*-coordinate as each other. Does the relationship between the *y*-intercept and the *y*-coordinates of *A* and *B* remain the same?

  yes

b. Set the *x*-coordinates of *A* and *B* to the same value and the *y*-coordinates to different values, as in Question 2(b).

- What does GeoGebra indicate as the *y*-intercept of the line?

  GeoGebra indicates that the *y*-intercept is "?."
- Why do you think this is true?

  Vertical lines do not cross the *y*-axis so there is no *y*-intercept. GeoGebra uses the symbol "?" for the coordinates of the *y*-intercept. During the class discussion, have a conversation with students to indicate that their answer should not be "?," but instead "no *y*-intercept." The only exception to this would be if both points are on the *y*-axis. In this case, every point on the line is a *y*-intercept.
- Change the *x*-coordinates of *A* and *B* so that the two points still have the same *x*-coordinate as each other. Does the line now have a *y*-intercept?

  no

Question 6:

Formulate a brief overview of the slope and *y*-intercept of a line. Write at least three facts about slope and *y*-intercept and be able to explain them to the class. Keep in mind that you should be able to explain as if your classmates have never learned about slope and *y*-intercept before. Your explanation may use what you have observed in all of the previous lessons.

The student responses will vary but there should be facts that include:

- the slope as a ratio of the rise over run
- the slope of the line will be negative if either the rise or run is negative and the other is positive
- the *y*-intercept is the intersection of the line and the *y*-axis.

Students might also mention specific situations where the slope is undefined or zero. Students should demonstrate a thorough knowledge of those main points.

Lesson 4: Applications of the Slope and *y*-Intercept of a Line

Time for Lesson: 45 minutes

Summary: Students will gain a better understanding of slope and *y*-intercept by using GeoGebra to visually solve applied problems. There are three application problems in this lesson and the handout will guide the students through a problem setup for each that leads to the solution. Students are expected to have some previous, non-GeoGebra, practice in writing equations of lines.

Materials:

- Guided student handout with application problems
- Device with GeoGebra
- Teacher notes with answer key

Lesson Procedures:

- Students will get their devices and the teacher will explain the lesson. (3 minutes)
- Students will work through the guided handout and interactive GeoGebra worksheet to complete application problems. (32 minutes)
    - The teacher will monitor students' progress and ask leading questions when appropriate.
    - As students solve each problem they will be asked pointed questions to build understanding.
    - Students will have to show the teacher their progress at certain checkpoints at the end of each problem to assure they are on the right track.
- The teacher will choose one or two students to present their GeoGebra worksheet to the class. (10 minutes)
- The teacher will assign homework.

Applications of the Slope and *y*-Intercept of a Line (Student Worksheet)

Directions: Each of the problems in this lesson describes a situation that can be modeled by a line. Create separate GeoGebra worksheets for each of the following three problems. Be prepared to share your findings with the class.

Problem 1: There is a 200-gallon tank that is partially filled with 30 gallons of water. Tess uses a hose to add more water to the tank at a rate of 10 gallons per minute. By following the steps below, we model this situation as a linear function. We then use this function to determine how much water is in the tank after Tess has been filling it for 12 minutes and how much time it will take to fill the tank completely.

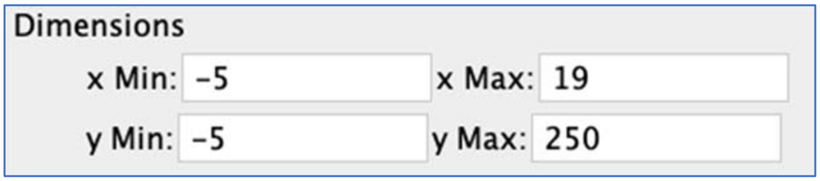
Important information from the problem:

There are two variables in this problem. The horizontal axis represents time in minutes since Tess started adding water and the vertical axis represents the total volume of water in the tank in gallons. We will use this information to choose appropriate scaling for the graph. First, answer the following question.

Question: Without solving the problem, estimate how long it will take Tess to fill the tank to its full capacity.

Part 1:

1. Open a new GeoGebra worksheet, click on the Settings button ⚙, and then click on Graphics. In the Basic tab, set the range for the *x*-axis to $[-5, 20]$. Also, change the *y*-axis range to $[-5, 250]$. See the screenshot below.



2. What is the initial volume of water in the tank?

3. What is the initial time value?

4. In the **Input:** bar, type $(0,30)$ to plot the initial point on the graph. This is the *y*-intercept of the line, since the *x*-coordinate is 0.

Part 2:

1. This problem requires you to find the total volume of water in the tank after different intervals of time. There were 30 gallons of water originally in the tank.
   a. One minute after Tess starts filling the tank, how much water will be in the tank?

   b. What point on the graph corresponds to this information?

   c. Plot this point on the graph by typing its coordinates in the Input bar.
   d. What part of the equation of a line does the rate of change of volume over time represent?

2. You now have two points on the graph. Use the **Line** tool to draw the line through the two points.
3. The problem asks for the volume of water in the tank after 12 minutes. Use the **Point on Object** tool to place a point on the line that would correspond to approximately 12 minutes. Record the coordinates of this point. (If necessary you

can read the coordinates of the point from the Algebra pane. Remember that you are estimating, so do not worry if the point is not exactly in the correct location.)

4. Describe how you can use this point to estimate the volume of water in the tank after 12 minutes.

5. Use the Point on Object tool to place a point on the line that would correspond to the full volume of the tank. Record the point below.

6. How can you use this point to estimate how long it would take Tess to fill the tank completely?

Answers to Problem 1:

Using your GeoGebra worksheet, graphically estimate the following:

a. How much water is in the tank after 12 minutes?

b. How long will it take to completely fill the tank?

c. For *x*-values beyond your answer to part (b), does the line represent the volume of water in the tank?

d. Explain why or why not.

Note: Do not discard this GeoGebra worksheet because you may need to share it at the end of class. Open a new GeoGebra worksheet in order to do the next problem.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next section.

Problem 2: An airplane 30,000 feet above the ground begins descending at a constant rate of 3,000 feet per minute. Write an equation to model the situation. Find the height of the plane above the ground after 7 minutes.

Important information from the problem:

a. What do the variables *x* and *y* represent in this problem?

b. Which variable will be represented on the horizontal axis?

c. Which variable will be represented on the vertical axis?

d. What information are we asked to find in this problem?

Part 1:

Similar to the first problem in this lesson, you must set the scale of the graph in the GeoGebra worksheet properly. A few things in this problem are different from the last problem though. In this problem, the plane is descending from an initial height.

1. Answer these questions:

   a. Is *y* increasing or decreasing as *x* increases?

   b. What is the initial height of the plane in the context of this problem?

c. Use your answer to part (b) to determine the range of the y-axis. Remember the upper end of the range should be slightly greater than your initial value so the point can be seen clearly. Also, the lower end should be below zero to see the x-axis.

2. In a new GeoGebra worksheet, set the range for the y-axis according to your answer in 1(c) above. Set the x-axis range to $[-1,12]$.

Part 2:

1. You must now place a point on the graph that corresponds the initial height of the plane. At the beginning of this problem, the height of the plane is 30,000 feet.
   a. What are the coordinates of the initial value point? Type these coordinates into the input bar to place that point on the graph.

   b. What does this point represent in the equation of the line?

2. After one minute, the plane has descended 3,000 feet.
   a. What will be the height of the plane after one minute?

   b. Use the Input bar to create a point that corresponds to the plane's height after one minute of descent. Record the coordinates below.

   c. What does the value 3,000 represent in this problem?

   d. Do you expect the slope of the line be positive or negative?

3. You now have two points on the graph. Use the 

tool to draw a line through them.

Answers to Problem 2:

Using your GeoGebra worksheet, graphically estimate the following:

a. This problem asks for the height of the plane after 7 minutes. Use the 

tool to place a point on the line that would correspond to 7 minutes. Record the coordinates.

b. What is the estimated height of the plane after 7 minutes?

c. How can you use the graph to determine when the plane lands?

d. How long will it take the plane to land if it maintains this constant descent?

Note: Do not discard this GeoGebra worksheet because you may need to share it at the end of class. Open a new GeoGebra worksheet in order to do the next problem.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next section.

Problem 3: Attorney Zachary charges an initial consultation fee of $850, and $150 for each hour of work after the initial consultation. Attorney Marcus charges $150 for the initial consultation, and $200 per hour after the initial consultation. Both attorneys charge on a continuous basis, so beyond the initial consultation, the client pays only for the time that the attorney actually works. Find how much you would pay each attorney for 10 hours of work. Which attorney has a lower overall fee for this amount of time? For 20 hours? At how many hours of work will these two attorneys have the same fee?

Important information from the problem:

a. What do the variables *x* and *y* represent in this problem?

b. Which variable will be represented on the horizontal axis of the graph?

c. Which variable will be represented on the vertical axis of the graph?

d. What information are we asked to find in this problem?

Note: Keep in mind that there are two separate attorneys in this problem. This means there will be two lines on the GeoGebra worksheet.

Part 1:

1. You must first set an appropriate scale for the graph.
    a. What should the estimated range of the *x*-values be on the graph?

    b. What should the estimated range of the *y*-values be on the graph?

2. In a new GeoGebra worksheet, set the range for both axes according to your answers in Question 1.
3. Use the Input bar to enter the initial value point of Attorney Zachary's fee just as you did in the first two problems. Then calculate how much will be paid to Zachary after one hour of work and plot the corresponding point. Construct the line through these two points. Do the same for Attorney Marcus. Record these points below:
   - Attorney Zachary initial:
   - Attorney Zachary after one hour:
   - Attorney Marcus initial:
   - Attorney Marcus after one hour:

   a. Place a point on each line representing 10 hours of service.
      - What is the estimated fee for attorney Zachary for 10 hours of work?
      - What is the estimated fee for attorney Marcus for 10 hours of work?

   b. Place a point on each line representing 20 hours of service.
      - What is the estimated fee for attorney Zachary for 20 hours of work?
      - What is the estimated fee for attorney Marcus for 20 hours of work?

   c. Which attorney has a lower fee for 10 hours of work?

   d. Which attorney has a lower fee for 20 hours of work?

Part 2:

Use the Intersection tool to construct the intersection point.

1. Write the coordinates of this point. What does this point represent in the context of the problem?

2. After how many hours of work are the attorneys' fees equal?

Applications of the Slope and *y*-Intercept of a Line (Teacher Version)

Directions: Each of the problems in this lesson describes a situation that can be modeled by a line. Create separate GeoGebra worksheets for each of the following three problems. Be prepared to share your findings with the class.

Problem 1: There is a 200-gallon tank that is partially filled with 30 gallons of water. Tess uses a hose to add more water to the tank at a rate of 10 gallons per minute. By following the steps below, we model this situation as a linear function. We then use this function to determine how much water is in the tank after Tess has been filling it for 12 minutes and how much time it will take to fill the tank completely.

Important information from the problem:

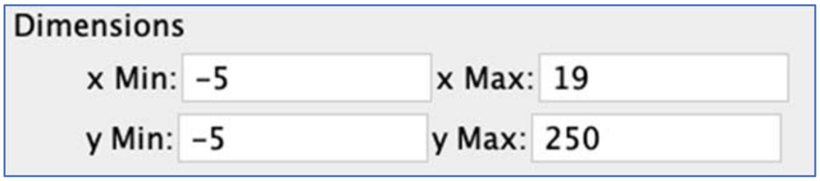
There are two variables in this problem. The horizontal axis represents time in minutes since Tess started adding water and the vertical axis represents the total volume of water in the tank in gallons. We will use this information to choose appropriate scaling for the graph. First, answer the following question.

Question: Without solving the problem, estimate how long it will take Tess to fill the tank to its full capacity.

> Answers will vary but students should guess a time that is no longer than 20 minutes since the tank is 200 gallons being filled at 10 gallons per minute. Therefore, after 20 minutes the tank would be overflowing.

Part 1:

1. Open a new GeoGebra worksheet, click on the Settings button, and then click on Graphics. In the Basic tab, set the range for the *x*-axis to $[-5, 20]$. Also, change the *y*-axis range to $[-5, 250]$. See the screenshot below.



2. What is the initial volume of water in the tank?

   30 gallons

3. What is the initial time value?

   0 minutes

4. In the **Input:** bar, type $(0,30)$ to plot the initial point on the graph. This is the *y*-intercept of the line, since the *x*-coordinate is 0.

Part 2:

1. This problem requires you to find the total volume of water in the tank after different intervals of time. There were 30 gallons of water originally in the tank.

   a. One minute after Tess starts filling the tank, how much water will be in the tank?

      40 gallons

   b. What point on the graph corresponds to this information?

      $(1,40)$

   c. Plot this point on the graph by typing its coordinates in the Input bar.

   d. What part of the equation of a line does the rate of change of volume over time represent?

      The slope of the line

2. You now have two points on the graph. Use the **Line** tool to draw the line through the two points.

3. The problem asks for the volume of water in the tank after 12 minutes. Use the **Point on Object** tool to place a point on the line that would correspond to approximately 12 minutes. Record the coordinates of this point. (If necessary you

can read the coordinates of the point from the Algebra pane. Remember that you are estimating, so do not worry if the point is not exactly in the correct location.)

The students should record points that are close to $(12,150)$.

4. Describe how you can use this point to estimate the volume of water in the tank after 12 minutes.

   The *y*-coordinate of this point would be an estimate of the amount of water in the tank after 12 minutes.

5. Use the Point on Object tool to place a point on the line that would correspond to the full volume of the tank. Record the point below.

   The students should record points that are close to $(17,200)$.

6. How can you use this point to estimate how long it would take Tess to fill the tank completely?

   The *x*-coordinate of this point will represent the time and the *y*-coordinate will represent the volume of water in the tank. Therefore, the volume is 200 when $y = 200$ and the corresponding *x*-coordinate will represent the time it takes for Tess to fill the tank completely.

Answers to Problem 1:

Using your GeoGebra worksheet, graphically estimate the following:

a. How much water is in the tank after 12 minutes?

   About 150 gallons

b. How long will it take to completely fill the tank?

   About 17 minutes

c. For *x*-values beyond your answer to part (b), does the line represent the volume of water in the tank?

   No.

d. Explain why or why not.

   Once the tank is filled, the water will overflow and the volume of the water in the tank will not increase.

Note: Do not discard this GeoGebra worksheet because you may need to share it at the end of class. Open a new GeoGebra worksheet in order to do the next problem.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next section.

Problem 2: An airplane 30,000 feet above the ground begins descending at a constant rate of 3,000 feet per minute. Write an equation to model the situation. Find the height of the plane above the ground after 7 minutes.

Important information from the problem:

a. What do the variables *x* and *y* represent in this problem?

   Feet above the ground (*y*)

   Time in minutes (*x*)

b. Which variable will be represented on the horizontal axis?

   Time in minutes (*x*)

c. Which variable will be represented on the vertical axis?

   Feet above the ground (*y*)

d. What information are we asked to find in this problem?

   The *y*-value when $x = 7$, which represents the height of the plane above the ground after 7 minutes.

Part 1:

Similar to the first problem in this lesson, you must set the scale of the graph in the GeoGebra worksheet properly. A few things in this problem are different from the last problem though. In this problem, the plane is descending from an initial height.

1. Answer these questions:

   a. Is *y* increasing or decreasing as *x* increases?

      Decreasing. The plane is getting closer to the ground; therefore the height above the ground is decreasing.

   b. What is the initial height of the plane in the context of this problem?

      30,000 feet

c. Use your answer to part (b) to determine the range of the *y*-axis. Remember the upper end of the range should be slightly greater than your initial value so the point can be seen clearly. Also, the lower end should be below zero to see the *x*-axis.

Range should be approximately $[-1000, 31000]$.

The upper bound should be greater than 30,000 and less than 35,000. In order to clearly see the points on the *x*-axis, the lower bound should be less than zero.

2. In a new GeoGebra worksheet, set the range for the *y*-axis according to your answer in 1(c) above. Set the *x*-axis range to $[-1,12]$.

Part 2:

1. You must now place a point on the graph that corresponds the initial height of the plane. At the beginning of this problem, the height of the plane is 30,000 feet.
   a. What are the coordinates of the initial value point? Type these coordinates into the input bar to place that point on the graph.

   $(0, 30000)$

   b. What does this point represent in the equation of the line?

   y-intercept

2. After one minute, the plane has descended 3,000 feet.
   a. What will be the height of the plane after one minute?

   27,000 feet

   b. Use the Input bar to create a point that corresponds to the plane's height after one minute of descent. Record the coordinates below.

   $(1, 27000)$

   c. What does the value 3,000 represent in this problem?

   The rate at which the plane is descending in feet per minute in this problem.

   d. Do you expect the slope of the line be positive or negative?

   The slope is negative because the second point is below and to the right of the first point. Another reason is that the height of the plane is decreasing.

3. You now have two points on the graph. Use the 

tool to draw a line through them.

Answers to Problem 2:

Using your GeoGebra worksheet, graphically estimate the following:

a. This problem asks for the height of the plane after 7 minutes. Use the 

tool to place a point on the line that would correspond to 7 minutes. Record the coordinates.

   The students should record points that are close to $(7,9000)$.

b. What is the estimated height of the plane after 7 minutes?

   About 9,000 feet

c. How can you use the graph to determine when the plane lands?

   The vertical axis represents the height of the plane above the ground. Therefore, when the line crosses the *x*-axis, the height of the plane is 0 feet, and thus the plane is on the ground. The *x*-coordinate of the *x*-intercept is the time when the plane is on the ground.

   Note: During the class discussion of this problem, be sure to have a conversation with the students about whether it is realistic to assume that the plane maintains a constant speed of descent until landing.

d. How long will it take the plane to land if it maintains this constant descent?

   10 minutes

Note: Do not discard this GeoGebra worksheet because you may need to share it at the end of class. Open a new GeoGebra worksheet in order to do the next problem.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next section.

Problem 3: Attorney Zachary charges an initial consultation fee of $850, and $150 for each hour of work after the initial consultation. Attorney Marcus charges $150 for the initial consultation, and $200 per hour after the initial consultation. Both attorneys charge on a continuous basis, so beyond the initial consultation, the client pays only for the time that the attorney actually works. Find how much you would pay each attorney for 10 hours of work. Which attorney has a lower overall fee for this amount of time? For 20 hours? At how many hours of work will these two attorneys have the same fee?

Important information from the problem:

a. What do the variables *x* and *y* represent in this problem?

   Time in hours (*x*)

   Cost of attorney services (*y*)

b. Which variable will be represented on the horizontal axis of the graph?

   Time in hours (*x*)

c. Which variable will be represented on the vertical axis of the graph?

   Cost of attorney services (*y*)

d. What information are we asked to find in this problem?

   The students are asked to compare the total cost of the two attorneys' services to find the less expensive option after 10 and 20 hours.

Note: Keep in mind that there are two separate attorneys in this problem. This means there will be two lines on the GeoGebra worksheet.

Part 1:

1. You must first set an appropriate scale for the graph.

   a. What should the estimated range of the *x*-values be on the graph?

      0 hours to at least 20 hours

   b. What should the estimated range of the *y*-values be on the graph?

      Minimum value less than 0, so the *x*-axis is visible

      Maximumvalue between 4,200 and 5,000

2. In a new GeoGebra worksheet, set the range for both axes according to your answers in Question 1.
3. Use the Input bar to enter the initial value point of Attorney Zachary's fee just as you did in the first two problems. Then calculate how much will be paid to Zachary after one hour of work and plot the corresponding point. Construct the line through these two points. Do the same for Attorney Marcus. Record these points below:
   - Attorney Zachary initial: $(0, 850)$
   - Attorney Zachary after one hour: $(1,1000)$
   - Attorney Marcus initial: $(0,150)$
   - Attorney Marcus after one hour: $(1, 350)$

   a. Place a point on each line representing 10 hours of service.
      - What is the estimated fee for attorney Zachary for 10 hours of work?
        $2,350
      - What is the estimated fee for attorney Marcus for 10 hours of work?
        $2,150

   b. Place a point on each line representing 20 hours of service.
      - What is the estimated fee for attorney Zachary for 20 hours of work?
        $3,850
      - What is the estimated fee for attorney Marcus for 20 hours of work?
        $4,150

   c. Which attorney has a lower fee for 10 hours of work?
      Attorney Marcus

   d. Which attorney has a lower fee for 20 hours of work?
      Attorney Zachary

Part 2:

Use the Intersection tool to construct the intersection point.

1. Write the coordinates of this point. What does this point represent in the context of the problem?

   $(14, 2950)$; The intersection point is where the attorney fees are equal.

2. After how many hours of work are the attorneys' fees equal?

   14 hours

Lesson 5: Addition and Subtraction of Functions

Time for Lesson: 45 minutes

Summary: The lesson is a graphical introduction to addition and subtraction of functions. Students will use a prepared GeoGebra worksheet and guided questions to investigate how to graphically add and subtract functions by plotting them in GeoGebra. The students will then make their own GeoGebra worksheet for more exploration and discovery. Students should be comfortable with the use of basic tools in GeoGebra. The prepared GeoGebra worksheet is included with the electronic copy of this essay.

Materials:

- Guided student handout with investigative questions
- Devices with GeoGebra
- Prepared GeoGebra file titled "Addition and Subtraction of Functions."
- Teacher notes with answer key

Lesson Procedures:

- Students will get their devices and the teacher will explain the lesson. (3 minutes)
- Students will complete the guided handout and interactive GeoGebra worksheet. (32 minutes)
    - The teacher will monitor students' progress and ask leading questions when appropriate.
    - As students complete each section of the guided handout, they will be asked pointed questions that develop a deeper understanding.
    - Students will have to show the teacher their progress at certain checkpoints before they can move on, to ensure they are on the right track.
- The teacher will choose one or two students to present their GeoGebra worksheet and investigative questions (10 minutes).
- The teacher will assign homework.

Addition and Subtraction of Functions (Student Worksheet)

Directions: Use this worksheet along with the GeoGebra file named "Addition and Subtraction of Functions" that has been shared with you. At the end of this activity, you will submit a new GeoGebra worksheet with the final problem on this handout.

Section 1: Adding functions symbolically

Just as we can add and subtract numbers, we can also add and subtract functions. If we start with two functions f and g, we can add or subtract them to get new functions $f+g$ and $f-g$. We define these new functions by

$$(f+g)(x) = f(x)+g(x) \quad \text{and} \quad (f-g)(x) = f(x)-g(x).$$

Example A:

Let $f(x) = x^2 + 2x$ and $g(x) = 3x - 5$.

So $f(1) = 3$ and $g(1) = -2$.

Then, using the definition of $(f+g)(x)$,

$$\begin{aligned}(f+g)(1) &= f(1)+g(1)\\ &= 3+(-2)\\ &= 1.\end{aligned}$$

Example B:

Let $f(x) = x^2 + 2x$ and $g(x) = 3x - 5$.

Then, using the definition of $(f+g)(x)$,

$$\begin{aligned}(f+g)(x) &= (x^2+2x)+(3x-5)\\ &= x^2+5x-5.\end{aligned}$$

Understanding addition and subtraction of functions algebraically is important, but in this activity, you will investigate these operations graphically. Open the GeoGebra file titled "Addition and Subtraction of Functions." Use the GeoGebra worksheet to investigate the following problems one by one and be prepared to share your findings and responses at the end of the class period.

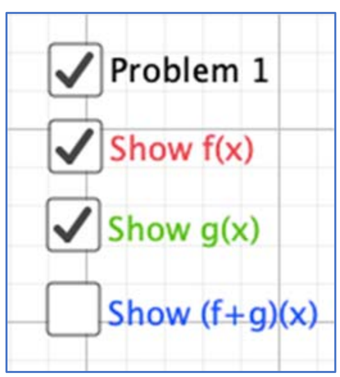
Note: Be sure to check with your teacher after each problem to ensure you are on the right track.

Section 2: Adding functions graphically

Instructions: The GeoGebra file is set up for the three problems you will be working on in this lesson. It is clearly marked for each problem. While you are working on Problem 1, you will only see the functions for that problem. When you move to Problem 2, the other functions will be hidden. After you click a check box for a particular problem, you can click to check a box for a function in order to see the corresponding graph. Click again to uncheck to box and hide the graph. Follow the lesson carefully to explore how addition and subtraction of functions can be performed graphically.

Problem 1:

1. Check the box for Problem 1 and make sure that only the boxes for $f(x)$ and $g(x)$ are checked.



2. Recall that if the point (x,y) is on the graph of f, then $y=f(x)$. So $f(1)$ is the y-coordinate of the point on the graph of f whose x-coordinate is 1.
   a. Use the graph to find the value of $f(1)$.

   b. Use the graph to find the value of $g(1)$.

3. The sum of f and g is the sum of the functions at each x-value. For example, $(f+g)(1)=f(1)+g(1)$.
   a. Using your answers from Question 2, what is $(f+g)(1)$?

   b. Use the Point tool in GeoGebra to place a point at $(1,(f+g)(1))$. Write its coordinates below.

   c. Once you have plotted the point, check the box "Show $(f+g)(x)$" to check whether the graph passes through this point.

4. Hide the graph of $(f+g)(x)$. Then, use the graphs of the functions $f(x)$ and $g(x)$ to find the indicated values of $(f+g)(x)$. Plot the corresponding points and record their coordinates. After you are finished, show the graph of $(f+g)(x)$ to see if your points are in the correct location. If a y-value is too great to fit in your current GeoGebra window, you must move the grid in the Graphics pane until you can plot the point. Always move the grid back so the origin is at the center of the screen.
   a. $(f+g)(-3)=$
   b. $(f+g)(0)=$
   c. $(f+g)(2)=$
   d. $(f+g)(-1)=$

5. Is $(f+g)(3)=(g+f)(3)$? What property of addition does this demonstrate?

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next problem.

Note: Delete the points you have placed on the graph for Problem 1 before moving on to Problem 2.

Section 3: Subtracting functions symbolically

Example C:

Let $f(x)=x^2-3$ and $g(x)=x+4$.

So $f(1)=-2$ and $g(1)=5$.

Then, using the definition of $(f-g)(x)$,

$$\begin{aligned}(f-g)(1)&=f(1)-g(1)\\&=-2-5\\&=-7.\end{aligned}$$

Example D:

Let $f(x)=x^2-3$ and $g(x)=x+4$.

Then, using the definition of $(f-g)(x)$,

$$\begin{aligned}(f-g)(x)&=(x^2-3)-(x+4)\\&=x^2-3-x-4\\&=x^2-x-7.\end{aligned}$$

Note: Before you start the next problem, uncheck the box for Problem 1 in the GeoGebra worksheet and check the box for Problem 2. Also, think about how the properties for addition and subtraction differ. The next two problems will focus on subtraction of functions.

Section 4: Subtracting functions graphically

Problem 2:

1. Show the graphs of $p(x)$ and $q(x)$ in the GeoGebra worksheet.
2. In this problem, we are going to focus on the difference $p(x)-q(x)$.
3. Just as with addition, we subtract two functions by subtracting their *y*-values at each *x*-value. In general, $(p-q)(x)=p(x)-q(x)$.
4. Practice this by finding the values of the following using the graphs:
   a. What is the value of $p(2)$?

   b. What is the value of $q(2)$?

   c. Find the difference $p(2)-q(2)$. This is the value of $(p-2)(2)$.

5. Using the Point tool in GeoGebra, place the point $(2,(p-q)(2))$ on the graph.
   a. Write its coordinates below.

   b. Check the box "Show $(p-q)(x)$" to check whether the graph passes through this point.
6. Hide the graph of the difference function $(p-q)(x)$ before starting the next step.
7. Find the following values from the graphs of the functions $p(x)$ and $q(x)$ and plot the corresponding points. Record the coordinates of the points. As in Problem 1, you may have to move the grid in the Graphics pane in order to plot the points. Remember to move the grid back so the origin is at the center of the screen. When you are done, use the graph of $(p-q)(x)$ to check your work.
   a. $(p-q)(-4)=$

b. $(p-q)(-2)=$

c. $(p-q)(0)=$

d. $(p-q)(2)=$

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next problem.

Note: Delete the points you have placed on the graph for Problem 2 before moving on to Problem 3. Also, uncheck the box for Problem 2 and check the box for Problem 3.

Problem 3:

In this problem you will continue to practice with the subtraction of functions.

1. Find the following values from the graphs of the functions $p(x)$ and $q(x)$ and plot the corresponding points.
2. Keep the graphs of $(s-t)(x)$ and $(t-s)(x)$ hidden until you are finished plotting each set of points. As in Problems 1 and 2, you may have to move the grid in the Graphics pane to plot the point. Remember to move the grid back so the origin is at the center of the screen.
3. Find the following values for $(s-t)(x)$ and plot the corresponding points. Record the coordinates of the points.

   a. $(s-t)(-2)=$

   b. $(s-t)(0)=$

   c. $(s-t)(2)=$

   d. $(s-t)(4)=$

   e. $(s-t)(-4)=$

4. Show the graph of $(s-t)(x)$ and verify that the points you plotted are correct. Then delete the points you have created on the graph of $(s-t)(x)$.

5. Find the following values for $(t-s)(x)$, and graph the corresponding points. Record the coordinates of the points.
   a. $(t-s)(-2)=$
   b. $(t-s)(0)=$
   c. $(t-s)(2)=$
   d. $(t-s)(4)=$
   e. $(t-s)(-4)=$
6. Show the graph of and $(t-s)(x)$ and verify that the points you plotted are correct.
7. Compare and contrast the graphs of $(s-t)(x)$ and $(t-s)(x)$.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next problem.

Section 5: Investigating further

Problem 4:

Open a blank GeoGebra worksheet and graph the following functions and points. When you are finished, submit the worksheet to your teacher.

1. Graph the following functions by typing them into the input bar.
   a. $f(x)=2x^2-3$
   b. $g(x)=2x^2+2x-1$
2. Use the graphs of f and g to find the values of the following and plot the corresponding points. Record the coordinates of the points.

a. $(f+g)(-2)=$

b. $(f+g)(-1)=$

c. $(f+g)(0)=$

d. $(f+g)(1)=$

e. $(f+g)(2)=$

3. Use the graphs of f and g to find the values of the following and plot the corresponding points in a different color than you used in Question 2.

a. $(f-g)(-2)=$

b. $(f-g)(-1)=$

c. $(f-g)(0)=$

d. $(f-g)(1)=$

e. $(f-g)(2)=$

4. What type of function is $(f+g)(x)$?

5. What type of function is $(f-g)(x)$?

6. Use the formulas for f and g to explain why $(f-g)(x)$ is a different type of function than $(f+g)(x)$.

7. When you are finished, save the GeoGebra worksheet with the name “Problem 4” and make sure your name is on the GeoGebra file. Then, send it to your teacher electronically.

Addition and Subtraction of Functions (Teacher Version)

Directions: Use this worksheet along with the GeoGebra file named "Addition and Subtraction of Functions" that has been shared with you. At the end of this activity, you will submit a new GeoGebra worksheet with the final problem on this handout.

Section 1: Adding functions symbolically

Just as we can add and subtract numbers, we can also add and subtract functions. If we start with two functions f and g, we can add or subtract them to get new functions $f+g$ and $f-g$. We define these new functions by

$$(f+g)(x) = f(x)+g(x) \quad \text{and} \quad (f-g)(x) = f(x)-g(x).$$

Example A:

Let $f(x)=x^2+2x$ and $g(x)=3x-5$.

So $f(1)=3$ and $g(1)=-2$.

Then, using the definition of $(f+g)(x)$,

$$\begin{aligned}(f+g)(1) &= f(1)+g(1)\\ &= 3+(-2)\\ &= 1.\end{aligned}$$

Example B:

Let $f(x)=x^2+2x$ and $g(x)=3x-5$.

Then, using the definition of $(f+g)(x)$,

$$\begin{aligned}(f+g)(x) &= \left(x^2+2x\right)+(3x-5)\\ &= x^2+5x-5.\end{aligned}$$

Understanding addition and subtraction of functions algebraically is important, but in this activity, you will investigate these operations graphically. Open the GeoGebra file titled "Addition and Subtraction of Functions." Use the GeoGebra worksheet to investigate the following problems one by one and be prepared to share your findings and responses at the end of the class period.

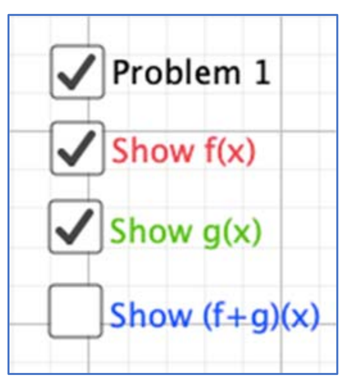
Note: Be sure to check with your teacher after each problem to ensure you are on the right track.

Section 2: Adding functions graphically

Instructions: The GeoGebra file is set up for the three problems you will be working on in this lesson. It is clearly marked for each problem. While you are working on Problem 1, you will only see the functions for that problem. When you move to Problem 2, the other functions will be hidden. After you click a check box for a particular problem, you can click to check a box for a function in order to see the corresponding graph. Click again to uncheck to box and hide the graph. Follow the lesson carefully to explore how addition and subtraction of functions can be performed graphically.

Problem 1:

1. Check the box for Problem 1 and make sure that only the boxes for $f(x)$ and $g(x)$ are checked.



2. Recall that if the point (x, y) is on the graph of f, then $y = f(x)$. So $f(1)$ is the y-coordinate of the point on the graph of f whose x-coordinate is 1.
   a. Use the graph to find the value of $f(1)$.

      $f(1) = 1$

   b. Use the graph to find the value of $g(1)$.

      $g(1) = 2$

3. The sum of f and g is the sum of the functions at each x-value. For example, $(f+g)(1) = f(1) + g(1)$.
   a. Using your answers from Question 2, what is $(f+g)(1)$?

      $(f+g)(1) = 3$

   b. Use the Point tool in GeoGebra to place a point at $(1, (f+g)(1))$. Write its coordinates below.

      $(1, 3)$

   c. Once you have plotted the point, check the box "Show $(f+g)(x)$" to check whether the graph passes through this point.

4. Hide the graph of $(f+g)(x)$. Then, use the graphs of the functions $f(x)$ and $g(x)$ to find the indicated values of $(f+g)(x)$. Plot the corresponding points and record their coordinates. After you are finished, show the graph of $(f+g)(x)$ to see if your points are in the correct location. If a y-value is too great to fit in your current GeoGebra window, you must move the grid in the Graphics pane until you can plot the point. Always move the grid back so the origin is at the center of the screen.
   a. $(f+g)(-3) = 9 + (-6) = 3$, point $(-3, 3)$
   b. $(f+g)(0) = 0 + 0 = 0$, point $(0, 0)$
   c. $(f+g)(2) = 4 + 4 = 8$, point $(2, 8)$
   d. $(f+g)(-1) = 1 + (-2) = -1$, point $(-1, -1)$

5. Is $(f+g)(3)=(g+f)(3)$? What property of addition does this demonstrate?

Yes; this is the commutative property.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next problem.

Note: Delete the points you have placed on the graph for Problem 1 before moving on to Problem 2.

Section 3: Subtracting functions symbolically

Example C:

Let $f(x)=x^2-3$ and $g(x)=x+4$.

So $f(1)=-2$ and $g(1)=5$.

Then, using the definition of $(f-g)(x)$,

$$\begin{aligned}(f-g)(1) &= f(1)-g(1)\\ &= -2-5\\ &= -7.\end{aligned}$$

Example D:

Let $f(x)=x^2-3$ and $g(x)=x+4$.

Then, using the definition of $(f-g)(x)$,

$$\begin{aligned}(f-g)(x) &= (x^2-3)-(x+4)\\ &= x^2-3-x-4\\ &= x^2-x-7.\end{aligned}$$

Note: Before you start the next problem, uncheck the box for Problem 1 in the GeoGebra worksheet and check the box for Problem 2. Also, think about how the properties for addition and subtraction differ. The next two problems will focus on subtraction of functions.

Section 4: Subtracting functions graphically

Problem 2:

1. Show the graphs of $p(x)$ and $q(x)$ in the GeoGebra worksheet.
2. In this problem, we are going to focus on the difference $p(x)-q(x)$.
3. Just as with addition, we subtract two functions by subtracting their *y*-values at each *x*-value. In general, $(p-q)(x)=p(x)-q(x)$.
4. Practice this by finding the values of the following using the graphs:
   a. What is the value of $p(2)$? $p(2)=1$
   b. What is the value of $q(2)$? $q(2)=-2$
   c. Find the difference $p(2)-q(2)$. This is the value of $(p-2)(2)$.
      $p(2)-q(2)=3$
5. Using the Point tool in GeoGebra, place the point $(2,(p-q)(2))$ on the graph.
   a. Write its coordinates below.
      $(2,3)$
   b. Check the box "Show $(p-q)(x)$" to check whether the graph passes through this point.
6. Hide the graph of the difference function $(p-q)(x)$ before starting the next step.
7. Find the following values from the graphs of the functions $p(x)$ and $q(x)$ and plot the corresponding points. Record the coordinates of the points. As in Problem 1, you may have to move the grid in the Graphics pane in order to plot the points. Remember to move the grid back so the origin is at the center of the screen. When you are done, use the graph of $(p-q)(x)$ to check your work.
   a. $(p-q)(-4)=4-4=0$, point $(-4,0)$

b. $(p-q)(-2)=1-2=-1$, point $(-2,-1)$

c. $(p-q)(0)=0-0=0$, point $(0,0)$

d. $(p-q)(2)=1-(-2)=3$, point $(2,3)$

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next problem.

Note: Delete the points you have placed on the graph for Problem 2 before moving on to Problem 3. Also, uncheck the box for Problem 2 and check the box for Problem 3.

Problem 3:

In this problem you will continue to practice with the subtraction of functions.

1. Find the following values from the graphs of the functions $p(x)$ and $q(x)$ and plot the corresponding points.
2. Keep the graphs of $(s-t)(x)$ and $(t-s)(x)$ hidden until you are finished plotting each set of points. As in Problems 1 and 2, you may have to move the grid in the Graphics pane to plot the point. Remember to move the grid back so the origin is at the center of the screen.
3. Find the following values for $(s-t)(x)$ and plot the corresponding points. Record the coordinates of the points.

   a. $(s-t)(-2)=2-(-1)=3$, point $(-2,3)$

   b. $(s-t)(0)=0-1=-1$, point $(0,-1)$

   c. $(s-t)(2)=2-3=-1$, point $(2,-1)$

   d. $(s-t)(4)=8-5=3$, point $(4,3)$

   e. $(s-t)(-4)=8-(-3)=11$, point $(-4,11)$
4. Show the graph of $(s-t)(x)$ and verify that the points you plotted are correct. Then delete the points you have created on the graph of $(s-t)(x)$.

5. Find the following values for $(t-s)(x)$, and graph the corresponding points. Record the coordinates of the points.

   a. $(t-s)(-2) = -1-2=-3$, point $(-2,-3)$

   b. $(t-s)(0) = 1-0=1$, point $(0,1)$

   c. $(t-s)(2) = 3-2=1$, point $(2,1)$

   d. $(t-s)(4) = 5-8=-3$, point $(4,-3)$

   e. $(t-s)(-4) = -3-8=-11$, point $(-4,-11)$

6. Show the graph of and $(t-s)(x)$ and verify that the points you plotted are correct.

7. Compare and contrast the graphs of $(s-t)(x)$ and $(t-s)(x)$.

   - Both cross the *x*-axis at the same points.
   - The *y*-intercepts of the graphs are different.
   - The graphs of $(s-t)(x)$ and $(t-s)(x)$ are reflections of each other over the *x*-axis.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next problem.

Section 5: Investigating further

Problem 4:

Open a blank GeoGebra worksheet and graph the following functions and points. When you are finished, submit the worksheet to your teacher.

1. Graph the following functions by typing them into the input bar.

   a. $f(x) = 2x^2 - 3$

   b. $g(x) = 2x^2 + 2x - 1$

2. Use the graphs of *f* and *g* to find the values of the following and plot the corresponding points. Record the coordinates of the points.

a. $(f+g)(-2)=5+3=8$, point $(-2,8)$

b. $(f+g)(-1)=-1+(-1)=-2$, point $(-1,-2)$

c. $(f+g)(0)=-3+(-1)=-4$, point $(0,-4)$

d. $(f+g)(1)=-1+3=2$, point $(1,2)$

e. $(f+g)(2)=5+11=16$, point $(2,16)$

3. Use the graphs of f and g to find the values of the following and plot the corresponding points in a different color than you used in Question 2.

a. $(f-g)(-2)=5-3=2$, point $(-2,2)$

b. $(f-g)(-1)=-1-(-1)=0$, point $(-1,0)$

c. $(f-g)(0)=-3-(-1)=-2$, point $(0,-2)$

d. $(f-g)(1)=-1-3=-4$, point $(1,-4)$

e. $(f-g)(2)=5-11=-6$, point $(2,-6)$

4. What type of function is $(f+g)(x)$?

A quadratic function

5. What type of function is $(f-g)(x)$?

A linear function

6. Use the formulas for f and g to explain why $(f-g)(x)$ is a different type of function than $(f+g)(x)$.

The x^2 terms cancel in the subtraction, so the resulting function is linear.

7. When you are finished, save the GeoGebra worksheet with the name "Problem 4" and make sure your name is on the GeoGebra file. Then, send it to your teacher electronically.

Lesson 6: Introduction to Scalar Multiples and Products of Functions

Time for Lesson: 45 minutes

Summary: Students will use a prepared GeoGebra worksheet and guided questions for a graphical investigation of the product of a function with a scalar or another function. The students will then make their own GeoGebra worksheet for more exploration and discovery. Students should be comfortable with the use of basic tools in GeoGebra. The prepared GeoGebra worksheet is included with the electronic copy of this essay.

Materials:

- Guided student handout with investigative questions
- Devices with GeoGebra
- Prepared GeoGebra file titled "Introduction to Scalar Multiplication and Products of Functions."
- Teacher notes with answer key

Lesson Procedures:

- Students will get their devices and the teacher will explain the lesson. (3 minutes)
- Students will complete the guided handout and interactive GeoGebra worksheet. (32 minutes)
    - The teacher will monitor students' progress and ask leading questions when appropriate.
    - As students complete each section of the guided handout, they will be asked pointed questions that develop a deeper understanding.
    - Students will have to show the teacher their progress at certain checkpoints before they can move on, to ensure they are on the right track.
- The teacher will choose one or two students to present their GeoGebra worksheet. (10 minutes)
- The teacher will assign homework.

Introduction to Scalar Multiples and Products of Functions (Student Worksheet)

Directions: Use this worksheet along with the GeoGebra document "Introduction to Scalar Multiplication and Products of Functions." You will submit a new GeoGebra worksheet with the final problem on this handout.

Section 1: Multiplying functions symbolically

To multiply a function by a scalar, we multiply the output of the function by that scalar. For each number c and function f, we can create a new function $c \cdot f$ defined by $(c \cdot f)(x) = c \cdot \left[f(x) \right]$ for any value of x.

Example A:

Let $f(x) = x^2 + 2x$.

So $f(1) = 3$.

Then for $x = 1$, the value of $(4f)(x)$ is

$$\begin{aligned} (4f)(1) &= 4 \cdot f(1) \\ &= 4 \cdot 3 \\ &= 12. \end{aligned}$$

Example B:

Let $f(x) = x^2 + 2x$ and let $c = 3$.

Then

$$\begin{aligned} (cf)(x) &= c \cdot \left[f(x) \right] \\ &= 3\left(x^2 + 2x\right) \\ &= 3x^2 + 6x. \end{aligned}$$

Just as we can multiply numbers, we can also multiply functions. For any two functions *f* and *g*, we can form the product $f \cdot g$ defined by $(f \cdot g)(x) = f(x) \cdot g(x)$.

Example C:

Let $f(x) = x^2 + 2x$ and $g(x) = 3x - 5$.

So $f(1) = 3$ and $g(1) = -2$.

Then, using the definition of $(f \cdot g)(x)$,

$$\begin{aligned}(f \cdot g)(1) &= f(1) \cdot g(1) \\ &= 3 \cdot (-2) \\ &= -6.\end{aligned}$$

Example D:

Let $f(x) = x^2 + 2x$ and $g(x) = 3x - 5$.

Then, using the definition of $(f \cdot g)(x)$,

$$\begin{aligned}(f \cdot g)(x) &= f(x) \cdot g(x) \\ &= (x^2 + 2x) \cdot (3x - 5) \\ &= 3x^3 - 5x^2 + 6x^2 - 10x \\ &= 3x^3 + x^2 - 10x.\end{aligned}$$

Understanding how to multiply functions algebraically is important, but in this activity, you are going to investigate the process graphically. Open the GeoGebra worksheet titled "Introduction to Scalar Multiplication and Products of Functions." Use this GeoGebra page to investigate the following problems and be prepared to share your findings and responses at the end of the class.

Note: Be sure to check with your teacher after each problem to ensure you are on the right track.

Section 2: Multiplying functions graphically

Instructions: The GeoGebra file is set up for the three problems you will be working on in this lesson. It is clearly marked for each problem. While you are working on Problem 1, you will only see the functions for that problem. When you move to Problem 2, the other functions will be hidden. After you click a check box for a particular problem, you can click to check a box for a function in order to see the corresponding graph. Click again to uncheck to box and hide the graph. Follow the lesson carefully to explore how to graphically multiply a function by a scalar or another function.

Problem 1:

1. In this problem you will investigate the product of a function by two separate scalars.
2. Click the box for Problem 1 and make sure that only the graph of $f(x)$ is showing. For reference, see the illustration below:

3. To find the scalar product of a function by graphing you must first find the *y*-value of a point on the function for a given *x*-value.
    a. Place a point at $(1, f(1))$.
    b. Record the value of $f(1)$. This is the *y*-coordinate of the point you just plotted.

c. What is $\frac{1}{3} \cdot f(1)$?

d. Place a new point on the graph at $\left(1, \left(\frac{1}{3} \cdot f\right)(1)\right)$.

4. Repeat the process from Question 3 to find the values of the following and plot the corresponding points. Record the coordinates of the points. If a *y*-value is too great to fit in your current GeoGebra window, you must move the grid in the Graphics pane until you can plot the point. Always move the grid back so the origin is at the center of the screen.

   Let $c = \frac{1}{3}$.

   a. $c \cdot f(-2) =$

   b. $c \cdot f(-1) =$

   c. $c \cdot f(0) =$

   d. $c \cdot f(2) =$

   e. $c \cdot f(3) =$

5. Show the graph of $\frac{1}{3} f(x)$ and verify that the points you plotted are correct. Then delete the points you have created on the graph of $\frac{1}{3} f(x)$ and hide the graph of $\frac{1}{3} f(x)$.

6. Let $c = 3$. Find the values of the following and plot the corresponding points. Record the coordinates of the points.
   a. $c \cdot f(-2) =$
   b. $c \cdot f(-1) =$
   c. $c \cdot f(0) =$
   d. $c \cdot f(1) =$
   e. $c \cdot f(2) =$
7. After you plot the points above, check the box to show the graph of $3f(x)$. Make sure the graph goes through the points you plotted.
8. Show the graphs of both $\frac{1}{3}f(x)$ and $3f(x)$. Compare these to the graph of $f(x)$.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next problem.

Note: Delete the points you have placed on the graph before moving on to Problem 2. Also, uncheck the box for Problem 1 and check the box for Problem 2.

Problem 2:

1. Check the boxes for functions $p(x)$ and $q(x)$, so you can see their graphs.
2. In this problem, you are going to focus on the product $(p \cdot q)(x) = p(x) \cdot q(x)$.

3. In many ways, this is similar to addition and subtraction of functions because you multiply the outputs of the two functions to find the output of the function product. Use the graphs of p and q to find and record the answers below.
   a. What is the value of $p(2)$?
   b. What is the value of $q(2)$?
   c. Find $p(2)\cdot q(2)$.
   d. Plot the corresponding point on the worksheet. Do not show the graph of $(p\cdot q)(x)$ yet.
4. Find the values of the following and plot the corresponding points. Record the coordinates of the points. As in Problem 1, you may have to move the grid in the Graphics pane in order to plot the points. Remember to move the grid back so the origin is at the center of the screen.
   a. $p(-2)\cdot q(-2)=$
   b. $p(0)\cdot q(0)=$
   c. $p(4)\cdot q(4)=$
   d. $p(-4)\cdot q(-4)=$
5. Show the graph of $(p\cdot q)(x)$ and verify that the points you plotted are correct.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next section.

Note: Delete the points you have placed on the graph before moving on to Problem 3. Also, uncheck the box for Problem 2 and check the box for Problem 3.

Problem 3:

In Problem 3 you will do another example where you multiply two functions and then ultimately by a scalar.

1. Make sure you only have the graphs of functions $s(x)$ and $t(x)$ showing.
2. Use the graphs of s and t to find the values of the following and plot the corresponding points. Record the coordinates of the points.
   a. $(s \cdot t)(-4) =$
   b. $(s \cdot t)(-2) =$
   c. $(s \cdot t)(0) =$
   d. $(s \cdot t)(2) =$
   e. $(s \cdot t)(4) =$
3. Show the graph of $s \cdot t$ and verify that the points you plotted are correct. Then delete the points you have created on the graph of $s \cdot t$ and hide the graph of $s \cdot t$.
4. Use the graphs of s and t to find the values of the following and plot the corresponding points.
   a. $(t \cdot s)(-4) =$
   b. $(t \cdot s)(-2) =$
   c. $(t \cdot s)(0) =$
   d. $(t \cdot s)(2) =$
   e. $(t \cdot s)(4) =$
5. Show the graph of $t \cdot s$ and verify that the points you plotted are correct. Then delete the points you have created on the graph of $t \cdot s$ and hide the graph of $t \cdot s$.
6. Notice that $s \cdot t$ and $t \cdot s$ represent multiplication in different orders. Are the two functions different?

7. What product of multiplication justifies your answer to Question 6?

8. Use the five points that you recorded for $s \cdot t$ in Question 2 of Problem 3 to determine the corresponding points on the graph of $\frac{1}{4}(s \cdot t)$. Then plot these new points.

9. Compare the graphs of $s \cdot t$ and $\frac{1}{4}(s \cdot t)$.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next section.

Problem 4:

Open a blank GeoGebra worksheet.

1. Graph the following functions by typing them into the Input bar.
   a. $f(x) = x^2 - 4$
   b. $g(x) = -x^2 + 3$
2. Use the graphs of f and g to find the values of the following and plot the corresponding points. Record the coordinates of the points.
   a. $(f \cdot g)(-2) =$
   b. $(f \cdot g)(-1) =$
   c. $(f \cdot g)(0) =$
   d. $(f \cdot g)(1) =$
   e. $(f \cdot g)(2) =$

3. Check these points by typing $f \cdot g$ into the input bar.
4. What type of function is $f \cdot g$? How do you know?

5. When you are finished, save the GeoGebra worksheet with the name "Problem 4" and make sure your name is on the GeoGebra file. Then, send it to your teacher electronically.

Problem 5: Extra problem if time allows

Open a blank GeoGebra worksheet.

1. Type the equation for a function f (your choice) into the Input bar.
2. Multiply this function by the scalar -1 by typing $-1 \cdot f$ in the Input bar.
3. How does the graph of $-1 \cdot f$ compare to the graph of f?

4. Now, multiply f by a negative scalar c that is not -1.
5. How does the graph of $c \cdot f$ compare to the graph of f?

6. When you are finished, save the GeoGebra worksheet with the name "Problem 5" and make sure your name is in the GeoGebra file. Then send it to your teacher electronically.

Introduction to Scalar Multiples and Products of Functions (Teacher Version)

Directions: Use this worksheet along with the GeoGebra document "Introduction to Scalar Multiplication and Products of Functions." You will submit a new GeoGebra worksheet with the final problem on this handout.

Section 1: Multiplying functions symbolically

To multiply a function by a scalar, we multiply the output of the function by that scalar. For each number c and function f, we can create a new function $c \cdot f$ defined by $(c \cdot f)(x) = c \cdot [f(x)]$ for any value of x.

Example A:

Let $f(x) = x^2 + 2x$.

So $f(1) = 3$.

Then for $x = 1$, the value of $(4f)(x)$ is

$$\begin{aligned}(4f)(1) &= 4 \cdot f(1) \\ &= 4 \cdot 3 \\ &= 12.\end{aligned}$$

Example B:

Let $f(x) = x^2 + 2x$ and let $c = 3$.

Then

$$\begin{aligned}(cf)(x) &= c \cdot [f(x)] \\ &= 3(x^2 + 2x) \\ &= 3x^2 + 6x.\end{aligned}$$

Just as we can multiply numbers, we can also multiply functions. For any two functions *f* and *g*, we can form the product $f \cdot g$ defined by $(f \cdot g)(x) = f(x) \cdot g(x)$.

Example C:

Let $f(x) = x^2 + 2x$ and $g(x) = 3x - 5$.

So $f(1) = 3$ and $g(1) = -2$.

Then, using the definition of $(f \cdot g)(x)$,

$$\begin{aligned}(f \cdot g)(1) &= f(1) \cdot g(1) \\ &= 3 \cdot (-2) \\ &= -6.\end{aligned}$$

Example D:

Let $f(x) = x^2 + 2x$ and $g(x) = 3x - 5$.

Then, using the definition of $(f \cdot g)(x)$,

$$\begin{aligned}(f \cdot g)(x) &= f(x) \cdot g(x) \\ &= \left(x^2 + 2x\right) \cdot (3x - 5) \\ &= 3x^3 - 5x^2 + 6x^2 - 10x \\ &= 3x^3 + x^2 - 10x.\end{aligned}$$

Understanding how to multiply functions algebraically is important, but in this activity, you are going to investigate the process graphically. Open the GeoGebra worksheet titled “Introduction to Scalar Multiplication and Products of Functions.” Use this GeoGebra page to investigate the following problems and be prepared to share your findings and responses at the end of the class.

Note: Be sure to check with your teacher after each problem to ensure you are on the right track.

Section 2: Multiplying functions graphically

Instructions: The GeoGebra file is set up for the three problems you will be working on in this lesson. It is clearly marked for each problem. While you are working on Problem 1, you will only see the functions for that problem. When you move to Problem 2, the other functions will be hidden. After you click a check box for a particular problem, you can click to check a box for a function in order to see the corresponding graph. Click again to uncheck to box and hide the graph. Follow the lesson carefully to explore how to graphically multiply a function by a scalar or another function.

Problem 1:

1. In this problem you will investigate the product of a function by two separate scalars.
2. Click the box for Problem 1 and make sure that only the graph of $f(x)$ is showing. For reference, see the illustration below:

3. To find the scalar product of a function by graphing you must first find the *y*-value of a point on the function for a given *x*-value.
   a. Place a point at $(1, f(1))$.
   b. Record the value of $f(1)$. This is the *y*-coordinate of the point you just plotted.
   $f(1) = -3$

c. What is $\frac{1}{3} \cdot f(1)$?

$\frac{1}{3} \cdot f(1) = \frac{1}{3} \cdot (-3) = -1$

d. Place a new point on the graph at $\left(1, \left(\frac{1}{3} \cdot f\right)(1)\right)$.

$(1, -1)$

4. Repeat the process from Question 3 to find the values of the following and plot the corresponding points. Record the coordinates of the points. If a *y*-value is too great to fit in your current GeoGebra window, you must move the grid in the Graphics pane until you can plot the point. Always move the grid back so the origin is at the center of the screen.

Let $c = \frac{1}{3}$.

a. $c \cdot f(-2) = \frac{1}{3} \cdot (0) = 0$, point $(-2, 0)$

b. $c \cdot f(-1) = \frac{1}{3} \cdot (3) = 1$, point $(-1, 1)$

c. $c \cdot f(0) = \frac{1}{3} \cdot (0) = 0$, point $(0, 0)$

d. $c \cdot f(2) = \frac{1}{3} \cdot (0) = 0$, point $(2, 0)$

e. $c \cdot f(3) = \frac{1}{3} \cdot (15) = 5$, point $(3, 5)$

5. Show the graph of $\frac{1}{3} f(x)$ and verify that the points you plotted are correct. Then delete the points you have created on the graph of $\frac{1}{3} f(x)$ and hide the graph of $\frac{1}{3} f(x)$.

6. Let c = 3. Find the values of the following and plot the corresponding points. Record the coordinates of the points.

   a. $c \cdot f(-2) = 3 \cdot (0) = 0$, point $(-2, 0)$

   b. $c \cdot f(-1) = 3 \cdot (3) = 9$, point $(-1, 9)$

   c. $c \cdot f(0) = 3 \cdot (0) = 0$, point $(0, 0)$

   d. $c \cdot f(1) = 3 \cdot (-3) = -9$, point $(1, -9)$

   e. $c \cdot f(2) = 3 \cdot (0) = 0$, point $(2, 0)$

7. After you plot the points above, check the box to show the graph of $3f(x)$. Make sure the graph goes through the points you plotted.

8. Show the graphs of both $\frac{1}{3}f(x)$ and $3f(x)$. Compare these to the graph of $f(x)$.

   In comparison to the graph of $f(x)$, the graph of $3f(x)$ is stretched vertically away from the *x*-axis since the *y*-values are multiplied by a number greater than 1; and the graph of $\frac{1}{3}f(x)$ is compressed vertically towards the *x*-axis since the *y*-values are multiplied by a positive number less than 1.

   The graphs of all three functions have the same *x*-intercepts.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next problem.

Note: Delete the points you have placed on the graph before moving on to Problem 2. Also, uncheck the box for Problem 1 and check the box for Problem 2.

Problem 2:

1. Check the boxes for functions $p(x)$ and $q(x)$, so you can see their graphs.
2. In this problem, you are going to focus on the product $(p \cdot q)(x) = p(x) \cdot q(x)$.

3. In many ways, this is similar to addition and subtraction of functions because you multiply the outputs of the two functions to find the output of the function product. Use the graphs of *p* and *q* to find and record the answers below.
   a. What is the value of $p(2)$? $p(2)=1$
   b. What is the value of $q(2)$? $q(2)=0$
   c. Find $p(2)\cdot q(2)$. $p(2)\cdot q(2)=0$
   d. Plot the corresponding point on the worksheet. Do not show the graph of $(p\cdot q)(x)$ yet. point $(2,0)$
4. Find the values of the following and plot the corresponding points. Record the coordinates of the points. As in Problem 1, you may have to move the grid in the Graphics pane in order to plot the points. Remember to move the grid back so the origin is at the center of the screen.
   a. $p(-2)\cdot q(-2)=1\cdot(-2)=-2$, point $(-2,2)$
   b. $p(0)\cdot q(0)=0\cdot(-1)=-1$, point $(0,0)$
   c. $p(4)\cdot q(4)=4\cdot 1=4$, point $(4,4)$
   d. $p(-4)\cdot q(-4)=4\cdot(-3)=-12$, point $(4,-12)$
5. Show the graph of $(p\cdot q)(x)$ and verify that the points you plotted are correct.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next section.

Note: Delete the points you have placed on the graph before moving on to Problem 3. Also, uncheck the box for Problem 2 and check the box for Problem 3.

Problem 3:

In Problem 3 you will do another example where you multiply two functions and then ultimately by a scalar.

1. Make sure you only have the graphs of functions $s(x)$ and $t(x)$ showing.
2. Use the graphs of s and t to find the values of the following and plot the corresponding points. Record the coordinates of the points.
   a. $(s \cdot t)(-4) = 0 \cdot (-4) = 0$, point $(-4, 0)$
   b. $(s \cdot t)(-2) = -3 \cdot (-2) = 6$, point $(-2, 6)$
   c. $(s \cdot t)(0) = -4 \cdot 0 = 0$, point $(0, 0)$
   d. $(s \cdot t)(2) = -3 \cdot 2 = -6$, point $(2, -6)$
   e. $(s \cdot t)(4) = 0 \cdot 4 = 0$, point $(4, 0)$
3. Show the graph of $s \cdot t$ and verify that the points you plotted are correct. Then delete the points you have created on the graph of $s \cdot t$ and hide the graph of $s \cdot t$.
4. Use the graphs of s and t to find the values of the following and plot the corresponding points.
   a. $(t \cdot s)(-4) = -4 \cdot 0 = 0$, point $(-4, 0)$
   b. $(t \cdot s)(-2) = -2 \cdot (-3) = 6$, point $(-2, 6)$
   c. $(t \cdot s)(0) = 0 \cdot (-4) = 0$, point $(0, 0)$
   d. $(t \cdot s)(2) = 2 \cdot (-3) = -6$, point $(2, -6)$
   e. $(t \cdot s)(4) = 4 \cdot 0 = 0$, point $(4, 0)$
5. Show the graph of $t \cdot s$ and verify that the points you plotted are correct. Then delete the points you have created on the graph of $t \cdot s$ and hide the graph of $t \cdot s$.
6. Notice that $s \cdot t$ and $t \cdot s$ represent multiplication in different orders. Are the two functions different?

   They are not different.

7. What product of multiplication justifies your answer to Question 6?

   Multiplication of real numbers is commutative. Similarly, the multiplication of functions is commutative.

8. Use the five points that you recorded for $s \cdot t$ in Question 2 of Problem 3 to determine the corresponding points on the graph of $\frac{1}{4}(s \cdot t)$. Then plot these new points.

$(-4,0)$; $\left(-2, 1\frac{1}{2}\right)$; $(0,0)$; $\left(2, -1\frac{1}{2}\right)$; $(4,0)$

9. Compare the graphs of $s \cdot t$ and $\frac{1}{4}(s \cdot t)$.

In comparison to the graph of $s \cdot t$, the graph of $\frac{1}{4}(s \cdot t)$ is compressed vertically towards the *x*-axis since the *y*-values are multiplied by a positive number less than 1.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next section.

Problem 4:

Open a blank GeoGebra worksheet.

1. Graph the following functions by typing them into the Input bar.
   a. $f(x) = x^2 - 4$
   b. $g(x) = -x^2 + 3$
2. Use the graphs of f and g to find the values of the following and plot the corresponding points. Record the coordinates of the points.
   a. $(f \cdot g)(-2) = 0 \cdot -1 = 0$, point $(-2,0)$
   b. $(f \cdot g)(-1) = -3 \cdot 2 = -6$, point $(-1,-6)$
   c. $(f \cdot g)(0) = -4 \cdot 3 = -12$, point $(0,-12)$
   d. $(f \cdot g)(1) = -3 \cdot 2 = -6$, point $(1,-6)$
   e. $(f \cdot g)(2) = 0 \cdot -1 = 0$, point $(2,0)$

3. Check these points by typing $f \cdot g$ into the input bar.
4. What type of function is $f \cdot g$? How do you know?

   It is a 4^{th} degree polynomial because the product of the two x^2 terms is an x^4 term.
5. When you are finished, save the GeoGebra worksheet with the name "Problem 4" and make sure your name is on the GeoGebra file. Then, send it to your teacher electronically.

Problem 5: Extra problem if time allows

Open a blank GeoGebra worksheet.

1. Type the equation for a function f (your choice) into the Input bar.
2. Multiply this function by the scalar -1 by typing $-1 \cdot f$ in the Input bar.
3. How does the graph of $-1 \cdot f$ compare to the graph of f?

   The graphs are reflections of each other across the *x*-axis.
4. Now, multiply f by a negative scalar *c* that is not -1.
5. How does the graph of $c \cdot f$ compare to the graph of f?
   - If *c* is less than -1, the graph of $c \cdot f$ is a combination of a reflection of the graph of f across the *x*-axis and a vertical stretch from the *x*-axis.
   - If *c* is between -1 and 0, the graph of $c \cdot f$ is a combination of a reflection of the graph of f across the *x*-axis and a vertical compression towards the *x*-axis.

6. When you are finished, save the GeoGebra worksheet with the name "Problem 5" and make sure your name is in the GeoGebra file. Then send it to your teacher electronically.

Lesson 7: Introduction to Composition of Functions

Time for Lesson: 45 minutes

Summary: Students will use a prepared GeoGebra worksheet and guided questions for a graphical investigation of the composition of functions. The students will then make their own GeoGebra worksheet for more exploration and discovery. Students should be comfortable with the use of basic tools in GeoGebra. The prepared GeoGebra worksheet is included with the electronic copy of this essay.

Materials:

- Guided student handout with investigative questions
- Devices with GeoGebra
- Prepared GeoGebra file titled "Introduction to Composition of Functions"
- Teacher notes with answer key and important questions

Lesson Procedures:

- Students will get their devices and the teacher will explain the lesson. (3 minutes)
- Students will complete the guided handout and interactive GeoGebra worksheet. (32 minutes)
  - The teacher will monitor students' progress and ask leading questions when appropriate.
  - As students complete each section of the guided handout, they will be asked pointed questions that develop a deeper understanding.
  - Students will have to show the teacher their progress at certain checkpoints before they can move on, to ensure they are on the right track.
- The teacher will choose one or two teams to present their GeoGebra worksheet and investigative questions. (10 minutes)
- The teacher will assign homework.

Introduction to Composition of Functions (Student Worksheet)

Directions: Use this worksheet along with the GeoGebra document "Introduction to Composition of Functions." You will submit a new GeoGebra worksheet with the final problem on this handout.

Section 1: Composing functions symbolically

We compose two functions by applying one function to the result of another function. Given functions *f* and *g*, we define the composite function $f \circ g$ by

$$(f \circ g)(x) = f(g(x)).$$

Example:

Let $f(x) = x^2$ and $g(x) = 3x - 10$.

To compute $(f \circ g)(2)$ we first compute $g(2) = -4$.

Then

$$\begin{aligned}(f \circ g)(2) &= f(g(2)) \\ &= f(-4) \\ &= (-4)^2 \\ &= 16.\end{aligned}$$

In general, using the definition of $(f \circ g)(x)$,

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(3x - 10) \\ &= (3x - 10)^2.\end{aligned}$$

Understanding how to compose functions algebraically is important, but in this activity, you are going to investigate the process graphically. Open the GeoGebra worksheet titled "Introduction to Composition of Functions." Use this GeoGebra page to investigate the following problems and be prepared to share your findings and responses at the end of the class.

Note: Be sure to check with your teacher after each problem to ensure you are on the right track.

Section 2: Composing functions graphically

Instructions: The GeoGebra file is set up for the two problems you will be working on in this lesson. It is clearly marked for each problem. While you are working on Problem 1, you will only see the functions for that problem. When you move to Problem 2, the other functions will be hidden. After you click a check box for a particular problem, you can click to check a box for a function in order to see the corresponding graph. Click again to uncheck the box and hide the graph. Follow the lesson carefully to explore how to compose functions graphically.

Problem 1A:

1. In this problem you will investigate the composition of two separate functions.
2. Click the box for Problem 1 and make sure that only the graphs of $f(x)$ and $g(x)$ are showing.
3. Use the following steps to find $(f \circ g)(1)$ graphically.

    a. Place a point at $(1, g(1))$.

    b. Record the value of $g(1)$. This is the *y*-coordinate of the point you just plotted.

c. Plot the point $\left(g(1), f\left(g(1)\right)\right)$ on the graph of f and record its coordinates. The x-coordinate for this point is the value that you recorded in part (b) above.

d. Record the value of $(f \circ g)(1)$. This is the y-coordinate of the point in part (c) above.

4. Plot the point $\left(1, (f \circ g)(1)\right)$ and record its coordinates.

5. Use the procedure from Question 3 to find the following values and plot the corresponding points on the graph of $(f \circ g)(x)$ Record the coordinates of the points.
   a. $(f \circ g)(-2) =$
   b. $(f \circ g)(-1) =$
   c. $(f \circ g)(0) =$
   d. $(f \circ g)(1) =$
   e. $(f \circ g)(4) =$
6. Show the graph of $(f \circ g)(x)$ to make sure it goes through all of the points you plotted in Question 5.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next section.

Note: Delete the points you have placed on the graph before moving on to Problem 1B. Hide the graph of $(f \circ g)(x)$.

Problem 1B:

1. In this part of the lesson you are going to investigate the composite function $(g \circ f)(x)$.
2. Do you think that the graph of $(g \circ f)(x)$ will be the same as the graph of $(f \circ g)(x)$?
3. Use the following steps to find $(g \circ f)(2)$ graphically.
   a. Place a point at $(2, f(2))$.
   b. Record the value of $f(2)$. This is the *y*-coordinate of the point you just plotted.
   c. Plot the point $(f(2), g(f(2)))$ on the graph of *g* and record its coordinates. The *x*-coordinate for this point is the value that you recorded in part (b) above.
   d. Record the value of $(g \circ f)(2)$. This is the *y*-coordinate of the point in part (c) above.
4. Plot the point $(2, (g \circ f)(2))$ and record its coordinates.
5. Find the following values, plot the corresponding points, and record their coordinates.
   a. $(g \circ f)(-2) =$
   b. $(g \circ f)(-1) =$
   c. $(g \circ f)(0) =$
   d. $(g \circ f)(3) =$
6. Show the graph of $(g \circ f)(x)$ to check the points that you plotted in Question 5.

7. Are the functions $(g \circ f)(x)$ and $(f \circ g)(x)$ equal? Hint: Compare their graphs.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next problem.

Note: Delete the points that you plotted for Problem 1B before moving on to Problem 2. Uncheck the box for Problem 1 and check the box for Problem 2.

Problem 2:

1. Make sure that only the graphs of $r(x)$ and $q(x)$ are shown.
2. Find the following values, plot the corresponding points, and record their coordinates.
   a. $(r \circ q)(-2) =$
   b. $(r \circ q)(0) =$
   c. $(r \circ q)(6) =$
   d. $(r \circ q)(-6) =$
   e. $(r \circ q)(10) =$
3. Show the graph of $(r \circ q)(x)$ to make sure it goes through all of the points you plotted in Question 2.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next section.

Note: Delete the points you have placed on the graph before moving on to the next question. Hide the graph of $(r \circ q)(x)$.

4. Find the following values, plot the corresponding points, and record their coordinates.
   a. $(q \circ r)(-4) =$
   b. $(q \circ r)(-2) =$
   c. $(q \circ r)(0) =$
   d. $(q \circ r)(2) =$
   e. $(q \circ r)(4) =$
5. Show the graph of $(q \circ r)(x)$ to make sure it goes through all of the points you plotted in Question 4.
6. Are the functions $(r \circ q)(x)$ and $(q \circ r)(x)$ equal?

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next section.

Problem 3: Open a new GeoGebra window and graph the following functions. When you are finished, save the GeoGebra worksheet with the name "Problem 3" and make sure your name is on the GeoGebra file. Then send it to your teacher electronically.

1. Graph the following functions in GeoGebra.
   a. $f(x) = x^2 + 1$
   b. $g(x) = -x^2 - 1$
2. Use the graphs to find the values of the following and plot the corresponding points. Record the coordinates.
   a. $(f \circ g)(-2) =$
   b. $(f \circ g)(-1) =$
   c. $(f \circ g)(0) =$

d. $(f \circ g)(1) =$

e. $(f \circ g)(2) =$

3. Check these points by typing $f\left(g(x)\right)$ into the input bar.
4. Use the graphs to find the values of the following and plot the corresponding points in a different color than you used in Question 2. Record the coordinates.

   a. $(g \circ f)(-2) =$

   b. $(g \circ f)(-1) =$

   c. $(g \circ f)(0) =$

   d. $(g \circ f)(1) =$

   e. $(g \circ f)(2) =$

5. Check these points by typing $g\left(f(x)\right)$ into the input bar.
6. When you are finished, save the GeoGebra worksheet with the name "Problem 3" and make sure your name is in the GeoGebra file. Then send it to your teacher electronically.

Introduction to Composition of Functions (Teacher Version)

Directions: Use this worksheet along with the GeoGebra document "Introduction to Composition of Functions." You will submit a new GeoGebra worksheet with the final problem on this handout.

Section 1: Composing functions symbolically

We compose two functions by applying one function to the result of another function. Given functions *f* and *g*, we define the composite function $f \circ g$ by

$$(f \circ g)(x) = f(g(x)).$$

Example:

Let $f(x) = x^2$ and $g(x) = 3x - 10$.

To compute $(f \circ g)(2)$ we first compute $g(2) = -4$.

Then

$$\begin{aligned}(f \circ g)(2) &= f(g(2)) \\ &= f(-4) \\ &= (-4)^2 \\ &= 16.\end{aligned}$$

In general, using the definition of $(f \circ g)(x)$,

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(3x-10) \\ &= (3x-10)^2.\end{aligned}$$

Understanding how to compose functions algebraically is important, but in this activity, you are going to investigate the process graphically. Open the GeoGebra worksheet titled "Introduction to Composition of Functions." Use this GeoGebra page to investigate the following problems and be prepared to share your findings and responses at the end of the class.

Note: Be sure to check with your teacher after each problem to ensure you are on the right track.

Section 2: Composing functions graphically

Instructions: The GeoGebra file is set up for the two problems you will be working on in this lesson. It is clearly marked for each problem. While you are working on Problem 1, you will only see the functions for that problem. When you move to Problem 2, the other functions will be hidden. After you click a check box for a particular problem, you can click to check a box for a function in order to see the corresponding graph. Click again to uncheck the box and hide the graph. Follow the lesson carefully to explore how to compose functions graphically.

Problem 1A:

1. In this problem you will investigate the composition of two separate functions.
2. Click the box for Problem 1 and make sure that only the graphs of $f(x)$ and $g(x)$ are showing.
3. Use the following steps to find $(f \circ g)(1)$ graphically.
   a. Place a point at $(1, g(1))$.
   b. Record the value of $g(1)$. This is the *y*-coordinate of the point you just plotted.
      $g(1) = 0$

c. Plot the point $\left(g(1), f\left(g(1)\right)\right)$ on the graph of f and record its coordinates. The *x*-coordinate for this point is the value that you recorded in part (b) above.

$(0,0)$

d. Record the value of $(f \circ g)(1)$. This is the *y*-coordinate of the point in part (c) above.

$(f \circ g)(1) = 0$

4. Plot the point $\left(1, (f \circ g)(1)\right)$ and record its coordinates.

$(1,0)$

5. Use the procedure from Question 3 to find the following values and plot the corresponding points on the graph of $(f \circ g)(x)$ Record the coordinates of the points.

a. $(f \circ g)(-2) = 9$, point $(-2,9)$

b. $(f \circ g)(-1) = 4$, point $(-1,4)$

c. $(f \circ g)(0) = 1$, point $(0,1)$

d. $(f \circ g)(2) = 1$, point $(2,1)$

e. $(f \circ g)(4) = 9$, point $(4,9)$

6. Show the graph of $(f \circ g)(x)$ to make sure it goes through all of the points you plotted in Question 5.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next section.

Note: Delete the points you have placed on the graph before moving on to Problem 1B. Hide the graph of $(f \circ g)(x)$.

Problem 1B:

1. In this part of the lesson you are going to investigate the composite function $(g \circ f)(x)$.
2. Do you think that the graph of $(g \circ f)(x)$ will be the same as the graph of $(f \circ g)(x)$?

   The students will discover the answer in Question 7 below.
3. Use the following steps to find $(g \circ f)(2)$ graphically.
   a. Place a point at $(2, f(2))$.

   b. Record the value of $f(2)$. This is the *y*-coordinate of the point you just plotted.

      $f(2) = 4$

   c. Plot the point $(f(2), g(f(2)))$ on the graph of *g* and record its coordinates. The *x*-coordinate for this point is the value that you recorded in part (b) above.

      $(4,3)$

   d. Record the value of $(g \circ f)(2)$. This is the *y*-coordinate of the point in part (c) above. $(g \circ f)(2) = 3$
4. Plot the point $(2, (g \circ f)(2))$ and record its coordinates.

   $(2,3)$
5. Find the following values, plot the corresponding points, and record their coordinates.
   a. $(g \circ f)(-2) = 3$, point $(-2,3)$

   b. $(g \circ f)(-1) = 0$, point $(-1,0)$

   c. $(g \circ f)(0) = -1$, point $(0,-1)$

   d. $(g \circ f)(3) = 8$, point $(3,8)$
6. Show the graph of $(g \circ f)(x)$ to check the points that you plotted in Question 5.

7. Are the functions $(g \circ f)(x)$ and $(f \circ g)(x)$ equal? Hint: Compare their graphs.

   The functions are not equal.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next problem.

Note: Delete the points that you plotted for Problem 1B before moving on to Problem 2. Uncheck the box for Problem 1 and check the box for Problem 2.

Problem 2:

1. Make sure that only the graphs of $r(x)$ and $q(x)$ are shown.
2. Find the following values, plot the corresponding points, and record their coordinates.
   a. $(r \circ q)(-2) = 2$, point $(-2, 2)$
   b. $(r \circ q)(0) = 0.5$, point $(0, 0.5)$
   c. $(r \circ q)(6) = 2$, point $(6, 2)$
   d. $(r \circ q)(-6) = 8$, point $(-6, 8)$
   e. $(r \circ q)(10) = 8$, point $(10, 8)$
3. Show the graph of $(r \circ q)(x)$ to make sure it goes through all of the points you plotted in Question 2.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next section.

Note: Delete the points you have placed on the graph before moving on to the next question. Hide the graph of $(r \circ q)(x)$.

4. Find the following values, plot the corresponding points, and record their coordinates.
   a. $(q \circ r)(-4) = -3$, point $(-4,-3)$
   b. $(q \circ r)(-2) = 0$, point $(-2,0)$
   c. $(q \circ r)(0) = 1$, point $(0,1)$
   d. $(q \circ r)(2) = 0$, point $(2,0)$
   e. $(q \circ r)(4) = -3$, point $(4,-3)$
5. Show the graph of $(q \circ r)(x)$ to make sure it goes through all of the points you plotted in Question 4.
6. Are the functions $(r \circ q)(x)$ and $(q \circ r)(x)$ equal?

   The functions are not equal.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next section.

Problem 3: Open a new GeoGebra window and graph the following functions. When you are finished, save the GeoGebra worksheet with the name "Problem 3" and make sure your name is on the GeoGebra file. Then send it to your teacher electronically.

1. Graph the following functions in GeoGebra.
   a. $f(x) = x^2 + 1$
   b. $g(x) = -x^2 - 1$
2. Use the graphs to find the values of the following and plot the corresponding points. Record the coordinates.
   a. $(f \circ g)(-2) = 26$, point $(-2,26)$
   b. $(f \circ g)(-1) = 5$, point $(-1,5)$
   c. $(f \circ g)(0) = 2$, point $(0,2)$

d. $(f \circ g)(1) = 5$, point $(1, 5)$

e. $(f \circ g)(2) = 26$, point $(2, 26)$

3. Check these points by typing $f(g(x))$ into the input bar.

4. Use the graphs to find the values of the following and plot the corresponding points in a different color than you used in Question 2. Record the coordinates.

a. $(g \circ f)(-2) = -26$, point $(-2, -26)$

b. $(g \circ f)(-1) = -5$, point $(-1, -5)$

c. $(g \circ f)(0) = -2$, point $(0, -2)$

d. $(g \circ f)(1) = -5$, point $(1, -5)$

e. $(g \circ f)(2) = -26$, point $(2, -26)$

5. Check these points by typing $g(f(x))$ into the input bar.

6. When you are finished, save the GeoGebra worksheet with the name "Problem 3" and make sure your name is in the GeoGebra file. Then send it to your teacher electronically.

Lesson 8: Parallel, Intersecting and Coincident Lines

Time for Lesson: 45 minutes

Summary: Students will use GeoGebra to investigate different configurations of three lines in a plane.

Materials:

- Guided student handout with investigative questions.
- Devices with GeoGebra
- Prepared GeoGebra worksheet titled "Parallel, Intersecting and Coincident Lines"
- Teacher notes with answer key

Lesson Procedures:

- Students will get their devices and the teacher will explain the lesson. (3 minutes)
- Students will complete the guided handout while using the interactive GeoGebra worksheet. (32 minutes)
  - The teacher will monitor students' progress and ask leading questions where appropriate.
  - Students will have to show the teacher their progress at certain checkpoints before they can move on, to ensure they are on the right track.
- The teacher will choose one or two students to present their findings to the class. (10 minutes)
- The teacher will assign homework.

Parallel, Intersecting and Coincident Lines (Student Worksheet)

Recall the following definitions:

- A <u>point</u> is a location in a plane.
- A <u>line</u> passes through two points, is straight and extends in both directions indefinitely.
- <u>Parallel lines</u> are lines in a plane that never meet or intersect.
- <u>Intersecting lines</u> are two lines that share exactly one point.
- <u>Coincident lines</u> are two lines that lie on top of one another. They are technically the same line.

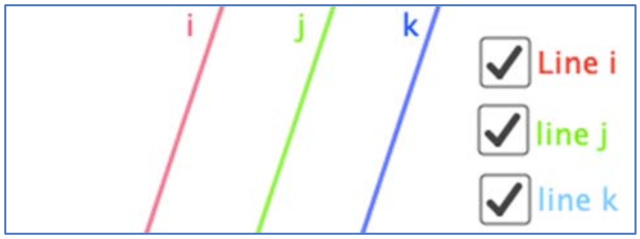
GeoGebra Note: When GeoGebra returns "?" as the result of a construction, it means the result does not exist. For example, if GeoGebra returns "?" when finding the point of intersection of two lines, it means the lines do not intersect.

Problem: Describe all possible configurations of three lines in a plane. You will use the definitions above and a GeoGebra worksheet to manipulate lines and help you solve this problem.

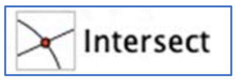
Directions: Open the GeoGebra worksheet "Parallel, Intersecting and Coincident Lines" and follow the guided handout below to solve the problem. There are several parts to this problem so check with your teacher at the indicated checkpoints to make sure you are on the right track.

Part 1:

1. Once the GeoGebra worksheet is open, make sure the boxes for lines *i*, *j* and *k* are checked as in the image below.



2. What appears to be the relationship between lines *i*, *j*, and *k*? Why?

3. Use the 

tool to find the intersection between each pair of lines and record the result from GeoGebra below.
   a. Intersection point of lines *i* and *j*:
   b. Intersection point of lines *j* and *k*:
   c. Intersection point of lines *k* and *i*:
4. Are your answers in Question 3 consistent with your prediction in Question 2?

5. Look at the equations of the lines. How are the slopes of the three lines related?

6. Are the *y*-intercepts of the lines be the same?

7. Drag the three lines to different positions in the graphics pane. Do the answers to Questions (6) and (7) change?

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving on to the next section.

Part 2:

1. Hide line *k* and show line *g*.
2. Use the Intersect tool to find the intersection points of any two of the lines *i*, *j* and *g*. How many intersection points did you find?
3. What is the relationship of line *g* to lines *i* and *j*?
4. Is the slope of line *g* the same as the slope of lines *i* and *j* ?
5. If two lines intersect at a single point, is it possible that they have the same slope?
6. Drag the two parallel lines to different positions in the graphics pane. Which aspects of the lines change and which do not change?
7. What happens if you drag line *g* to a new location?

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next section.

Part 3:

1. Show lines *f*, *g* and *h*, and hide lines *i* and *j*.
2. Change the equations if necessary:
    - line *g:* $y = -2x + 1$
    - line *h:* $y = 4x + 1$
    - line *f:* $y = 2x + 1$

3. Use the Intersect tool to find the coordinates of the intersection points for each pair of lines.
    a. Intersection of lines *g* and *f:*
    b. Intersection of lines *f* and *h:*
    c. Intersection of lines *h* and *g:*
4. What do the intersection points have in common?

5. How many intersection points are there all together?

6. Compare and contrast the equations of these three lines.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving on to the next section.

Part 4:

1. Without making any changes, predict how many intersection points there will be if you drag line *g* to a new location.

2. Drag line *g* to a new location. How many intersection points are there now all together?

3. Drag lines *g*, *h* and *f* to new locations.
    a. Can there be more than three intersection points all together?

    b. Why or why not?

c. Is it possible to drag lines *g*, *h* and *f* so that there are only two intersection points? Why or why not? (Note that intersection points may be off the screen.)

d. Is it possible to change the equations in the input boxes so that there are exactly two intersection points?

4. Reset the equations to those given at the beginning of Part 3.
5. Given three lines with different slopes, how many distinct intersection points can they have?

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving on to the next section.

Part 5:

1. Show lines *i*, *j* and *k*, and hide all other lines.
2. In their associated input boxes, make sure the equations are:
   - line *i:* $y = 3x - 2$
   - line *j:* $y = 3x + 1$
   - line *k:* $y = 3x + 1$
3. What do you notice about lines *j* and *k*?

4. How many different lines are visible?

5. Based on the definitions given at the beginning of this handout, what type of lines are *j* and *k*?

6. Use the input box to change the equation for line *i*: $y = 3x + 1$.
7. What is the relationship between the equations of all three lines?

8. How many different lines are visible?

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving on to the next section.

Part 6: Based on what we have discovered, answer the following questions.

1. What do we know about the slope and *y*-intercept of lines that are intersecting?

2. Compare and contrast lines that are parallel.

3. Compare and contrast lines that are coincident.

4. Make a list of all possible configurations of three lines in a plane. Be prepared to share this with the class so we can compile a full list. You can use the GeoGebra worksheet to continue investigating if you want.

Parallel, Intersecting and Coincident Lines (Teacher Version)

Recall the following definitions:

- A point is a location in a plane.
- A line passes through two points, is straight and extends in both directions indefinitely.
- Parallel lines are lines in a plane that never meet or intersect.
- Intersecting lines are two lines that share exactly one point.
- Coincident lines are two lines that lie on top of one another. They are technically the same line.

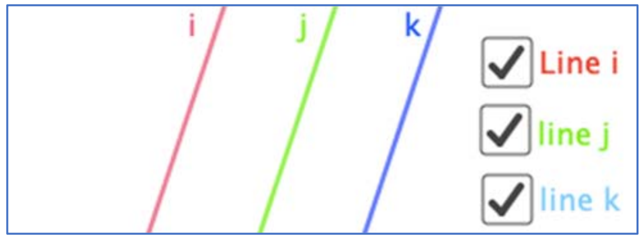
GeoGebra Note: When GeoGebra returns "?" as the result of a construction, it means the result does not exist. For example, if GeoGebra returns "?" when finding the point of intersection of two lines, it means the lines do not intersect.

Problem: Describe all possible configurations of three lines in a plane. You will use the definitions above and a GeoGebra worksheet to manipulate lines and help you solve this problem.

Directions: Open the GeoGebra worksheet "Parallel, Intersecting and Coincident Lines" and follow the guided handout below to solve the problem. There are several parts to this problem so check with your teacher at the indicated checkpoints to make sure you are on the right track.

Part 1:

1. Once the GeoGebra worksheet is open, make sure the boxes for lines i, j and k are checked as in the image below.



2. What appears to be the relationship between lines *i*, *j*, and *k*? Why?

   The lines appear to be parallel.

   Parallel lines do not intersect and the lines *i*, *j*, and *k* do not appear to intersect.

3. Use the 

tool to find the intersection between each pair of lines and record the result from GeoGebra below.

   a. Intersection point of lines *i* and *j*: none or ?

   b. Intersection point of lines *j* and *k*: none or ?

   c. Intersection point of lines *k* and *i*: none or ?

4. Are your answers in Question 3 consistent with your prediction in Question 2?

   With no intersection points, the lines are in fact parallel.

5. Look at the equations of the lines. How are the slopes of the three lines related?

   The lines have the same slope.

6. Are the *y*-intercepts of the lines be the same?

   No, they are all different.

7. Drag the three lines to different positions in the graphics pane. Do the answers to Questions (6) and (7) change?

   The slopes of the lines stay the same. The *y*-intercepts change but the *y*-intercepts of the three lines are still different from each other. If the students get two lines to be coincident, then the slope and *y*-intercepts will be the same. This will be covered later in the lesson.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving on to the next section.

Part 2:

1. Hide line *k* and show line *g*.
2. Use the Intersect tool to find the intersection points of any two of the lines *i*, *j* and *g*. How many intersection points did you find?

   Two intersection points
3. What is the relationship of line *g* to lines *i* and *j*?

   Line *g* intersects line *i* and *j*.
4. Is the slope of line *g* the same as the slope of lines *i* and *j* ?

   The slope of line *g* is different from the slope of *i* and *j*.
5. If two lines intersect at a single point, is it possible that they have the same slope?

   No. If the lines have the same steepness, they would have to have more than one point in common.
6. Drag the two parallel lines to different positions in the graphics pane. Which aspects of the lines change and which do not change?

   The intersection points with *g* change. The slopes of each line remain the same and the *y*-intercepts of the lines change.
7. What happens if you drag line *g* to a new location?

   Line *g* still intersects lines *i* and *j*, but at different points.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving to the next section.

Part 3:

1. Show lines *f*, *g* and *h*, and hide lines *i* and *j*.
2. Change the equations if necessary:
   - line *g:* $y = -2x + 1$
   - line *h:* $y = 4x + 1$
   - line *f:* $y = 2x + 1$

3. Use the Intersect tool to find the coordinates of the intersection points for each pair of lines.
   a. Intersection of lines *g* and *f:* $(0,1)$
   b. Intersection of lines *f* and *h:* $(0,1)$
   c. Intersection of lines *h* and *g:* $(0,1)$
4. What do the intersection points have in common?
   They are the same point.
5. How many intersection points are there all together?
   One intersection point.
6. Compare and contrast the equations of these three lines.
   The important point is that all three lines have different slopes. For this particular example, the lines have the same *y*-intercept because they intersect at the *y*-axis.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving on to the next section.

Part 4:

1. Without making any changes, predict how many intersection points there will be if you drag line *g* to a new location.
   Students should predict that there will be three intersection points.
2. Drag line *g* to a new location. How many intersection points are there now all together?
   Three separate intersection points.
3. Drag lines *g*, *h* and *f* to new locations.
   a. Can there be more than three intersection points all together?
      No
   b. Why or why not?
      Each pair of lines can intersect at most once and there are 3 pairs of lines. So there can be a maximum of 3 separate intersection points.

c. Is it possible to drag lines g, h and f so that there are only two intersection points? Why or why not? (Note that intersection points may be off the screen.)

No. Dragging the lines does not change their slopes. So if there are two intersection points, then one of the lines must intersect both of the other lines in different places. Since those other two lines have different slopes, they must also intersect.

d. Is it possible to change the equations in the input boxes so that there are exactly two intersection points?

Yes. If two lines have the same slope and the third line has a different slope, then there will be two intersection points.

4. Reset the equations to those given at the beginning of Part 3.
5. Given three lines with different slopes, how many distinct intersection points can they have?

   Three lines with different slopes will intersect at exactly one point or exactly three points.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving on to the next section.

Part 5:

1. Show lines i, j and k, and hide all other lines.
2. In their associated input boxes, make sure the equations are:
   - line i: $y = 3x - 2$
   - line j: $y = 3x + 1$
   - line k: $y = 3x + 1$
3. What do you notice about lines j and k?

   They are the same line and have the same equation.
4. How many different lines are visible?

   Only two lines are visible because lines j and k are identical.

5. Based on the definitions given at the beginning of this handout, what type of lines are *j* and *k*?

   Coincident lines.

6. Use the input box to change the equation for line *i*: $y = 3x + 1$.

7. What is the relationship between the equations of all three lines?

   They are all the same.

8. How many different lines are visible?

   One line.

CHECKPOINT: Check with your teacher to make sure you are on the right track before moving on to the next section.

Part 6: Based on what we have discovered, answer the following questions.

1. What do we know about the slope and *y*-intercept of lines that are intersecting?

   Intersecting lines have different slopes. Intersecting lines will have different *y*-intercepts unless the intersection point is on the *y*-axis.

2. Compare and contrast lines that are parallel.

   Lines that are parallel have the same slope. They have different *y*-intercepts.

3. Compare and contrast lines that are coincident.

   Coincident lines have the same equation and the lines are the same.

4. Make a list of all possible configurations of three lines in a plane. Be prepared to share this with the class so we can compile a full list. You can use the GeoGebra worksheet to continue investigating if you want.

   - Three parallel lines
   - Two parallel lines and one line that intersects both of them
   - Three intersecting lines with a common intersection point
   - Three intersecting lines with three intersection points
   - Three coincident lines
   - Two coincident lines and one line that is parallel to them
   - Two coincident lines that one line that intersects them

Conclusion

Algebra 1 is a course that lays the foundation for the future of any math student. The topic of linear functions and their properties can be found in math classes throughout high school and any post-secondary curriculum. Because of this, students must have a solid understanding of these topics. As a teacher, I have tried to find different ways to present these topics in my classes. GeoGebra has been a great platform to teach algebraic topics in a more visual manner.

The eight lessons in this essay use GeoGebra to complement some topics in the Algebra 1 curriculum. These lessons can be used as reinforcement after the topics are taught traditionally. They can also be used as differentiation for specific learning groups within a class. I have had anecdotal success using them as lessons with my own students.

GeoGebra is a platform I use regularly, and as a math teacher I believe it is a successful complement to my lessons. Many students can use GeoGebra to build a deeper cognitive understanding of certain math topics. If used properly, technology can be a teacher's greatest asset and can revolutionize the way we teach our students, which will allow them to grow as mathematicians.

Works Cited

Dietiker, L. (2013). *College Preparatory Mathematics 1: Algebra 1* (Vol. 5). Sacramento, CA: CPM Educational Program.

Kanold, T. D., Burger, E. B., Dixon, J. K., Larson, M. R., & Leinwand, S. (2015). *Integrated Mathematics 1*. Orlando, FL: Houghton Mifflin Harcourt.

https://www.geogebra.org