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INTRODUCTION OF INFINITE SERIES IN HIGH SCHOOL LEVEL CALCULUS

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The essay of Ericka L. Bella is hereby accepted:

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Introduction

For the past six years, I have taught a variety of calculus courses, ranging from AP Calculus BC, College Credit Plus Calculus 1 and 2, and Calculus, which is an introductory course that does not lead to college credit. The curriculum in each of these courses differs significantly. While one course starts the year with limits and differentiation, another slowly builds up to calculus topics, beginning with a hearty review of pre-calculus material. Depending on which calculus course students decide to take, they may end the year with derivatives and integration of trigonometric functions, while others are learning various techniques of integration.

Regardless of what calculus course my students decide to take, I want them to be as well prepared as possible for the mathematics classes they may take in college. For most of my students, calculus is the last mathematics class they will take in high school and I believe exposing them to new content is essential, even if we cannot cover it in depth due to time restrictions. Many of my past students have come back to visit while in college and mentioned that learning calculus topics in college has been significantly easier because they have had some exposure to these ideas in high school.

For this essay, I have designed an introduction to sequences and series for my non-college level Calculus course. This course begins with precalculus material and ends with the calculus of trigonometric functions. The students in this course do not encounter any content regarding sequences or series in the current curriculum. Based on my previous experiences as a student, entering college calculus without having learned about the convergence and divergence of series was a challenging task. There are a multitude of tests and rules to learn and knowing when and how to apply each can be confusing.

This introductory unit includes the following for each lesson: teacher notes, guided notes sheets, and an activity sheet, with solutions printed in red. Blank student documents are available in the appendix. I used language, notation, and organization of the sequences and series material as in the textbook *Calculus of a Single Variable* by Larson, Hostetler, and Edwards [2].

Due to the daily schedule at my school, I have planned each lesson for an 80-minute class period. I will lead students through the guided notes, with students completing the notes as reference for future use. Students will work in groups of two or three to complete the activity for the lesson. Students will provide their own technology for each lesson, as all students in my courses are required to own a TI-Nspire CX calculator.

Since there are no Ohio Mathematics Learning Standards for Calculus, I have linked standards for each activity to the AP Calculus BC College Board course description.

Additionally, all activities reflect the following four mathematical practices as defined by College Board [1].

> Mathematical Practice 1: Determine expressions and values using mathematical procedures and rules.
>
> Mathematical Practice 2: Translate mathematical information from a single representation or across multiple representations.
>
> Mathematical Practice 3: Justify reasoning and solutions.
>
> Mathematical Practice 4: Use correct notation, language, and mathematical conventions to communicate results or solutions.

Overall, I hope to expose my high school calculus students to the foundation of sequences and series. This is not intended to be an all-encompassing study, but instead an introduction on which they can later build. Although there are many tests for determining the convergence of series, I have purposefully selected six in particular to focus on: The n^{th}-Term Test, Geometric Series, p-Series, Direct Comparison Test, Limit Comparison Test, and Ratio Test. Students do not learn partial fraction decomposition or infinite integrals in this course. Therefore, I have decided to leave Telescoping Series and the Integral Test out of this unit.

Lesson 1: Introduction to Sequences and Series

<u>Teacher Notes</u>

Overview:

In this lesson, students will be introduced to sequences and series. Although some students enter calculus with prior knowledge of these terms, most students need a refresher to jog their memories. Although sequences and series are discussed in Algebra 2, the focus is often only arithmetic and geometric sequences and series. While these types of sequences and series are revisited in calculus, arithmetic and geometric sequences and series are very simplistic compared to what students see in a calculus course. This introduction is a great time to review factorials as well.

In this first activity, students will classify sequences and series. Students will be given sequences and asked to list terms, as well as find an expression for the n^{th} term of a sequence. Students will predict whether a sequence will have a limit given the pattern for the n^{th} term. Finally, students will be asked to find partial sums of a series.

When creating the examples for this activity, I wanted to focus on sequences and series that combined different types of expressions, as well as a mixture of positive and negative terms. It has been my experience that students especially struggle with finding a pattern for the n^{th} term when the expressions for the first few terms have been simplified, so I decided to include some examples of this scenario as well.

Objectives:

- Identify sequences and series
- List the terms of a sequence
- Write a formula for the n^{th} term of a sequence
- Determine whether a sequence converges or diverges
- Calculate partial sums of a series

Standards:

II. Limits: LIM-7.A.1: The n^{th} partial sum is defined as the sum of the first n terms of a series.

Guided Notes with Answers

Introduction to Sequences and Series Name: ________________________

Definitions

Sequence: a function whose domain is the set of positive integers; for the purpose of this course, a sequence will refer to an infinite list of numbers that often follows a pattern

We use subscript notation to represent the terms of the sequence: $a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots$

The n^{th} term is denoted by a_n.

The entire sequence is denoted by $\{a_n\}$.

Series: the sum of the terms of a sequence

We can calculate partial sums as well as some infinite sums.

Partial sums are denoted by S_m, where the first m terms of the sequence are added.

Notation using sigma: $S_m = \sum_{n=1}^{m} a_n = a_1 + a_2 + a_3 + \cdots + a_m$

If the limit L of a sequence exists as n goes to infinity, then the sequence converges to L. If the limit of a sequence does not exist as n goes to infinity, then the sequence diverges.

Example 1: Write the first 5 terms of the sequence whose n^{th} term is $a_n = \dfrac{3^n}{n!}$.

$a_1 = \dfrac{3^1}{1!} = \dfrac{3}{1} = 3$ $\quad a_2 = \dfrac{3^2}{2!} = \dfrac{9}{2}$ $\quad a_3 = \dfrac{3^3}{3!} = \dfrac{27}{6} = \dfrac{9}{2}$ $\quad a_4 = \dfrac{3^4}{4!} = \dfrac{81}{24} = \dfrac{27}{8}$

$a_5 = \dfrac{3^5}{5!} = \dfrac{243}{120} = \dfrac{81}{40}$

The first five terms of the sequence are $3,\ \dfrac{9}{2},\ \dfrac{9}{2},\ \dfrac{27}{8},\ \dfrac{81}{40}$.

Example 2: Find an expression for the n^{th} term of the sequence $3,\ 7,\ 11,\ 15,\ \ldots$.

$a_n = 4n - 1$

Example 3: Find an expression for the n^{th} term of the sequence $1,\ -\frac{1}{2},\ \frac{1}{6},\ -\frac{1}{24},\ \ldots$.

$$a_n = \frac{(-1)^{n+1}}{n!}$$

Example 4: Determine if the sequence converges or diverges by finding the limit, if possible: $a_n = 5 - \frac{1}{n^3}$.

$$\lim_{n\to\infty}\left(5 - \frac{1}{n^3}\right) = 5$$

Since the limit exists, the sequence converges to 5.

Example 5: Determine if the sequence converges or diverges by finding the limit, if possible: $a_n = \cos\left(\frac{2}{n}\right)$.

$$\lim_{n\to\infty}\cos\left(\frac{2}{n}\right) = 1$$

Since the limit exists, the sequence converges to 1.

Example 6: Determine if the sequence converges or diverges by finding the limit, if possible: $a_n = 1 + (-1)^n$.

$\lim_{n\to\infty}\left[1 + (-1)^n\right]$ does not exist. The terms of this sequence alternate between 0 and 2.

The sequence diverges.

Example 7: Determine if the sequence converges or diverges by finding the limit, if possible: $a_n = \sin(\pi n)$.

If we consider the function $f(x) = \sin(\pi x)$ whose domain is the set of all real numbers, then $\lim_{n\to\infty}[f(x)] = \lim_{n\to\infty}[\sin(\pi x)]$ does not exist.

However, the domain of the sequence $g(n) = \sin(\pi n)$ is just the set of all positive integers. Therefore $g(n) = 0$ for every *n*, and thus $\lim_{n\to\infty}[g(n)] = \lim_{n\to\infty}[\sin(\pi n)] = 0$.

The sequence converges.

Example 8: Find the partial sum S_4 given $a_n = \dfrac{3n}{n+2}$.

$a_1 = \frac{3}{3} = 1$ $\quad a_2 = \frac{6}{4} = \frac{3}{2}$ $\quad a_3 = \frac{9}{5}$ $\quad a_4 = \frac{12}{6} = 2$

$S_4 = 1 + \frac{3}{2} + \frac{9}{5} + 2 = \frac{63}{10}$

Activity Sheet with Answers

Activity: Introduction to Sequences and Series　　Name: ________________________

1. Classify each of the following as a sequence or a series.

a) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

Sequence

b) $\frac{4}{5}+\frac{8}{125}+\frac{16}{625}+\frac{32}{3125}+\cdots$

Series

c) $\frac{1}{7}+\frac{2}{7}+\frac{6}{7}+\frac{24}{7}+\cdots$

Series

d) $-\frac{3}{8}, \frac{3}{16}, -\frac{3}{32}, \frac{3}{64}, \ldots$

Sequence

2. Create your own pattern for a sequence. Write the expression for the n^{th} term, using correct notation. Then list the first five terms of your sequence.

Answers will vary. Sample answer: $a_n = \left(\frac{1}{4}\right)^n$

$a_1 = \frac{1}{4}$　　$a_2 = \frac{1}{16}$　　$a_3 = \frac{1}{64}$　　$a_4 = \frac{1}{256}$　　$a_5 = \frac{1}{1024}$

3. For each of the following, write an expression for the n^{th} term of the sequence.

a) $-\frac{5}{3}, \frac{25}{3}, -\frac{125}{3}, \frac{625}{3}, -\frac{3125}{3}, \ldots$

$a_n = \frac{(-5)^n}{3}$

b) $27, \frac{27}{2}, \frac{9}{2}, \frac{9}{8}, \frac{9}{40}, \ldots$

$a_n = \frac{27}{n!}$

c) 1, 6, 11, 16, 21, ... $a_n = 5n - 4$

d) $-\frac{2}{7}, -\frac{6}{49}, -\frac{24}{343}, -\frac{120}{2401}, -\frac{720}{16807}, \ldots$ $a_n = -\frac{(n+1)!}{7^n}$

Find the first five terms of each of the following sequences. Then determine if the sequence converges or diverges. If the sequence converges, find its limit.

4. $a_n = \frac{5n}{3+n}$

$a_1 = \frac{5}{4}$ $a_2 = 2$ $a_3 = \frac{5}{2}$ $a_4 = \frac{20}{7}$ $a_5 = \frac{25}{8}$

The sequence converges: $\lim_{n\to\infty} \frac{5n}{3+n} = 5.$

5. $a_n = (-1)^n n!$

$a_1 = -1$ $a_2 = 2$ $a_3 = -6$ $a_4 = 24$ $a_5 = -120$

Notice that the absolute value of the terms is growing without bound and the terms alternate between negative and positive. The sequence diverges and the limit does not exist.

6. $a_n = \dfrac{5^n}{n^3}$

$a_1 = 5$ $\quad a_2 = \dfrac{25}{8}$ $\quad a_3 = \dfrac{125}{27}$ $\quad a_4 = \dfrac{625}{64}$ $\quad a_5 = 25$

Notice that the terms are growing without bound and this can be verified with L'Hospital's Rule. The sequence diverges and the limit does not exist.

7. Find the partial sum S_6 of the series $-4+(-2)+\left(-\frac{2}{3}\right)+\left(-\frac{1}{6}\right)+\cdots$

$a_n = \dfrac{-4}{n!}$

$a_1 = -4$ $\quad a_2 = -2$ $\quad a_3 = -\dfrac{2}{3}$ $\quad a_4 = -\dfrac{1}{6}$ $\quad a_5 = -\dfrac{1}{30}$ $\quad a_6 = -\dfrac{1}{180}$

$$S_7 = -4+(-2)+\left(-\frac{2}{3}\right)+\left(-\frac{1}{6}\right)+\left(-\frac{1}{30}\right)+\left(-\frac{1}{180}\right) = -\frac{1237}{180}$$

Lesson 2: The n^{th}-Term Test for Divergence

Teacher Notes

Overview:

Following the previous lesson, students review the topic of series at the start of this lesson. We then transition to learning about the first test we will study, the n^{th}-Term Test for Divergence. At this point, students should be reminded that we will focus mainly on series for the remainder of this unit.

In this activity, students will find the limit of the n^{th} term through two different methods. Students will find the limit of the n^{th} term by algebraic means as well as complete a table of values to approximate the limit of the n^{th} term. Students will make conclusions about whether a series diverges based on the value of this limit.

Based on my previous experience teaching this topic, I know students struggle with the conclusions that can accurately be made with the n^{th}-Term Test. Many of my students incorrectly conclude that a series converges if $\lim_{n\to\infty} a_n = 0$. I specifically designed the first few questions of this activity to get students thinking about what this test actually means and what conditions are needed to conclude that a series diverges with this test.

Objectives:

- Find the limit of the n^{th} term of a series
- Apply the n^{th}-Term Test for Divergence

Standards:

II. Limits: LIM-7.A.5: The n^{th}-Term Test is a test for divergence of a series.

Guided Notes with Answers

The n^{th}-Term Test for Divergence Name: ________________________

Recall from previous lesson

Series: the sum of the terms of a sequence

We can calculate partial sums as well as some infinite sums.

Notation using sigma: $S_m = \sum_{n=1}^{m} a_n = a_1 + a_2 + a_3 + \cdots + a_m$

Example 1: Consider the series $0+0+0+0+\cdots$.

If we let a_n represent the n^{th} term of this series, what is $\lim_{n\to\infty} a_n$?

$\lim_{n\to\infty} a_n = 0$

If we added infinitely many terms of this series, what do you think the sum would be?

Zero

Based on our answers above, do you think this series converges or diverges?

Converges

Example 2: Consider the series $1+\frac{1}{2}+\frac{1}{2}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\cdots$.

If we let a_n represent the n^{th} term of this series, what is $\lim_{n\to\infty} a_n$?

$\lim_{n\to\infty} a_n = 0$

If we added infinitely many terms of this series, what do you think the sum would be?

The sum would grow without bound. We can group terms to add to 1 and then add infinitely many 1's:

$$1+\left(\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)+\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right)+\cdots=1+1+1+1+\cdots \ .$$

Based on our answers above, do you think this series converges or diverges?

Diverges

Example 3: Consider the series $5+7+9+11+13+\cdots$.

If we let a_n represent the n^{th} term of this series, what is $\lim_{n\to\infty} a_n$?

$\lim_{n\to\infty} a_n = \infty$

Additionally, $\lim_{n\to\infty} a_n \neq 0$.

If we added infinitely many terms of this series, what do you think the sum would be?

The sum would grow without bound.

Based on our answers above, do you think this series converges or diverges?

Diverges

Example 4: Consider the series $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\cdots$.

If we let a_n represent the n^{th} term of this series, what is $\lim_{n\to\infty} a_n$?

$\lim_{n\to\infty} a_n = \frac{1}{3}$

Additionally, $\lim_{n\to\infty} a_n \neq 0$.

If we added infinitely many terms of this series, what do you think the sum would be?

The sum would grow without bound.

Based on our answers above, do you think this series converges or diverges?

Diverges

Based on our examples, what conclusion can we make regarding $\lim_{n\to\infty} a_n$ and the convergence or divergence of a series?

When $\lim_{n\to\infty} a_n \neq 0$, the series diverges.

This leads us to our first test to determine whether a series diverges.

Think about the following:

If a series converges, the limit of its n^{th} term must be 0.

That is, if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

It is generally easier to determine the limit of the n^{th} term than it is to determine whether a series converges. Therefore, we will use the contrapositive of the statement above.

The n^{th}-Term Test for Divergence

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

Note: If $\lim_{n\to\infty} a_n = 0$, we cannot make any conclusion about the series. Look back at **Example 1** and **Example 2** to verify this.

Additionally, notice that when we express a sum as $\sum a_n$ without bounds, we always mean an infinite sum, starting at a finite value of *n*. Since the series has infinitely many terms, the convergence or divergence of the series does not depend on the particular finite value of *n* we start with. However, the actual value of a convergent sum will depend on the initial value of *n*.

For example, let's look at $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \sum_{n=3}^{\infty} a_n$. Since a_1 and a_2 are finite, $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=3}^{\infty} a_n$ converges.

Example 5: Use the n^{th}-Term Test for Divergence to determine whether the series diverges.

$$\sum_{n=0}^{\infty} 5(1.23)^n$$

$a_n = 5(1.23)^n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 5(1.23)^n = \infty$

Therefore, $\lim_{n\to\infty} a_n \neq 0$ and $\sum_{n=0}^{\infty} 5(1.23)^n$ diverges.

Example 6: Use the n^{th}-Term Test for Divergence to determine whether the series diverges.

$$\sum_{n=3}^{\infty} \frac{7}{n(n+8)}$$

$a_n = \frac{7}{n(n+8)}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{7}{n(n+8)} = 0$

Since $\lim_{n\to\infty} a_n = 0$, we cannot make a conclusion about whether $\sum_{n=3}^{\infty} \frac{7}{n(n+8)}$ diverges. The n^{th}-Term Test is inconclusive for this series.

Example 7: Use the n^{th}-Term Test for Divergence to determine whether the series diverges.

$$\sum_{n=1}^{\infty} \frac{n}{2n+3}$$

$a_n = \frac{n}{2n+3}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2}$

Therefore, $\lim_{n\to\infty} a_n \neq 0$ and $\sum_{n=1}^{\infty} \frac{n}{2n+3}$ diverges.

Example 8: Use the n^{th}-Term Test for Divergence to determine whether the series diverges.

$$\sum_{n=1}^{\infty} \frac{3^n}{n!}$$

$a_n = \frac{3^n}{n!}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3^n}{n!} = 0$

Note: We can explain that $\lim_{n\to\infty} \frac{3^n}{n!} = 0$ using the "Law of Dominance". Although the numerator and denominator are growing without bound, factorials grow at a faster rate than exponentials. We have previously discussed this concept in the Calculus course.

Therefore for terms far out in the sequence, the denominator is much larger than the numerator, leading to $\lim_{n\to\infty} \frac{3^n}{n!} = 0$.

Since $\lim_{n\to\infty} a_n = 0$, we cannot make a conclusion about whether $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ diverges.

The n^{th}-Term Test is inconclusive for this series.

Example 9: Use the n^{th}-Term Test for Divergence to determine whether the series diverges.

$$\sum_{n=0}^{\infty}\left(9-\frac{1}{n^3}\right)$$

$a_n = 9-\frac{1}{n^3}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(9-\frac{1}{n^3}\right)=9$

Therefore, $\lim_{n\to\infty} a_n \neq 0$ and $\sum_{n=0}^{\infty}\left(9-\frac{1}{n^3}\right)$ diverges.

Activity Sheet with Answers

Activity: The n[th]*-Term Test for Divergence* Name:______________________

Determine if each of the following statements are True or False.

1. If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ must converge. False

2. If $\sum a_n$ diverges, then $\lim_{n\to\infty} a_n = 0$. False

3. If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. True

4. If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ must diverge. False

5. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ must converge. False

6. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ must diverge. True

7. If $\sum a_n$ diverges, then $\lim_{n\to\infty} a_n \neq 0$. False

8. If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n \neq 0$. False

Based on your results above, only statements 3 and 6 are true.

Therefore, if we can show that $\lim_{n\to\infty} a_n \neq 0$, then we can conclude that a series diverges using the n^{th}-Term Test for Divergence.

If we can show that $\lim_{n\to\infty} a_n = 0$, then the n^{th}-Term Test for Divergence is inconclusive.

9. Use a table to find $\lim_{n\to\infty} a_n$ given $a_n = \frac{8}{n+9}$. Round your answers to the nearest hundred thousandth.

n	1	10	100	1,000	10,000	100,000
a_n	0.8	0.42105	0.07339	0.00793	0.00080	0.00008

Based on your table, $\lim_{n\to\infty} a_n = 0$. What conclusion can we make about $\sum \frac{8}{n+9}$?

Since $\lim_{n\to\infty} a_n = 0$, the n^{th}-Term Test is inconclusive for this series. No conclusions can be made.

10. Use a table to find $\lim_{n\to\infty} a_n$ given $a_n = \frac{(n+1)!}{n!}$.

n	1	10	100	1,000	10,000	100,000
a_n	2	11	101	1,001	10,001	100,001

Based on your table, $\lim_{n\to\infty} a_n = \infty$. What conclusion can we make about $\sum \frac{(n+1)!}{n!}$?

Since $\lim_{n\to\infty} a_n \neq 0$, the n^{th}-Term Test can be used to conclude that $\sum \frac{(n+1)!}{n!}$ diverges.

For each of the following, use algebraic techniques to find $\lim_{n\to\infty} a_n$. Then apply the n^{th}-Term Test if possible.

11. $\sum_{n=1}^{\infty} \frac{7n}{3n-1}$

$a_n = \frac{7n}{3n-1}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{7n}{3n-1} = \frac{7}{3}$

Therefore, $\lim_{n\to\infty} a_n \neq 0$ and $\sum_{n=1}^{\infty} \frac{7n}{3n-1}$ diverges.

12. $\sum_{n=4}^{\infty} \frac{-3n+2}{4n^2-1}$

$a_n = \frac{-3n+2}{4n^2-1}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{-3n+2}{4n^2-1} = 0$

Since $\lim_{n\to\infty} a_n = 0$, we cannot make a conclusion about whether $\sum_{n=4}^{\infty} \frac{-3n+2}{4n^2-1}$ diverges.

The n^{th}-Term Test is inconclusive for this series.

13. $\sum_{n=0}^{\infty} 3\left(\frac{1}{4}\right)^n$

$a_n = 3\left(\frac{1}{4}\right)^n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 3\left(\frac{1}{4}\right)^n = 0$

Since $\lim_{n\to\infty} a_n = 0$, we cannot make a conclusion about whether $\sum_{n=0}^{\infty} 3\left(\frac{1}{4}\right)^n$ diverges.

The n^{th}-Term Test is inconclusive for this series.

14. $\sum_{n=2}^{\infty} \frac{-5n^3+2n}{8n^2+9n-1}$

$a_n = \frac{-5n^3+2n}{8n^2+9n-1}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{-5n^3+2n}{8n^2+9n-1} = -\infty$

Therefore, $\lim_{n\to\infty} a_n \neq 0$ and $\sum_{n=2}^{\infty} \frac{-5n^3+2n}{8n^2+9n-1}$ diverges.

15. $\sum_{n=1}^{\infty} 7$

$a_n = 7$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 7 = 7$

Therefore, $\lim_{n\to\infty} a_n \neq 0$ and $\sum_{n=1}^{\infty} 7$ diverges.

16. $\sum_{n=3}^{\infty}\left(-\frac{6}{n}+5\right)$

$a_n = -\frac{6}{n}+5$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(-\frac{6}{n}+5\right) = 5$

Therefore, $\lim_{n\to\infty} a_n \neq 0$ and $\sum_{n=3}^{\infty}\left(-\frac{6}{n}+5\right)$ diverges.

17. $\sum_{n=4}^{\infty}(-1)^n\left(\frac{3n}{5n-6}\right)$

$a_n = (-1)^n\left(\frac{3n}{5n-6}\right)$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(-1)^n\left(\frac{3n}{5n-6}\right)$ does not exist. As we go further out in the sequence, the terms alternate between values close to $\frac{3}{5}$ and values close to $-\frac{3}{5}$. Therefore, $\lim_{n\to\infty} a_n \neq 0$ and $\sum_{n=4}^{\infty}(-1)^n\left(\frac{3n}{5n-6}\right)$ diverges.

Lesson 3: Geometric Series

<u>Teacher Notes</u>

Overview:

In this lesson, students will learn about geometric series. Most students have already encountered this type of series in Algebra 2. However, Algebra 2 focuses on finite geometric series, while calculus extends to infinite geometric series.

Students begin by identifying geometric series. Next, students will find the common ratio of the terms of a geometric series and use this to determine whether the series converges. If the series converges, students will calculate the sum.

I have found that many students enjoy working with this type of series since it is rather straightforward. However, students stumble when given geometric series with an initial value of the index other than $n = 0$. For this reason, I included examples on both the guided notes and the activity sheet that have different initial index values.

Additionally, I wanted to include a few examples of series that may not appear to be geometric at first glance. To address this, I have included series where only the numerator or denominator is raised to the n^{th} power.

Objectives:

- Identify geometric series
- Determine the convergence of geometric series
- Calculate the sum of convergent geometric series

Standards:

II. Limits: LIM-7.A.3: A geometric series is a series with a constant ratio between successive terms.

II. Limits: LIM-7.A.4: If a is a real number and r is a real number such that $|r| < 1$, then the geometric series [converges and] $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$.

Guided Notes with Answers

Geometric Series Name: ________________________

Consider the series $3+\frac{3}{2}+\frac{3}{4}+\frac{3}{8}+\frac{3}{16}+\cdots$.

What do you notice about the terms of this series?

The terms have a constant ratio of $\frac{1}{2}$.

Series consisting of terms with a constant ratio are called geometric series. The terms of this type of series form a geometric sequence.

In general, the series given by $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots$ with $a \neq 0$ and $r \neq 0$ is a geometric series with common ratio *r*.

Example 1: Identify the ratio of the geometric series and write the series in sigma notation.

$5+10+20+40+80+\cdots$

The ratio is 2 and the series can be written as $\sum_{n=0}^{\infty} 5(2)^n$.

Example 2: Identify the ratio of the geometric series and write the series in sigma notation.

$\frac{8}{5}+\frac{2}{5}+\frac{1}{10}+\frac{1}{40}+\frac{1}{160}+\cdots$

The ratio is $\frac{1}{4}$ and the series can be written as $\sum_{n=0}^{\infty} \frac{8}{5}\left(\frac{1}{4}\right)^n$.

Example 3: Identify the ratio of the geometric series and write the series in sigma notation.

$1+(-3)+9+(-27)+81+\cdots$

The ratio is -3 and the series can be written as $\sum_{n=0}^{\infty} 1(-3)^n$ or $\sum_{n=0}^{\infty} (-3)^n$.

Identifying the ratio of a geometric series is a crucial skill because the ratio determines whether the series will converge or diverge. Let's investigate this below.

You may remember from Algebra 2 that there is a formula for the sum of the terms of a finite geometric sequence: $\sum_{n=0}^{m} ar^n = a\left(\frac{1-r^m}{1-r}\right)$ provided that $r \neq 1$.

If $r=1$, then $\sum_{n=0}^{m} ar^n = \sum_{n=0}^{m} a(1)^n = a(1)^0 + a(1)^1 + a(1)^2 + \cdots + a(1)^m = a(1+m)$.

We know that the value of an infinite series is the limit of its partial sums.

Using the formula above for $r \neq 1$, we can represent the m^{th} partial sum as $S_m = a\left(\frac{1-r^m}{1-r}\right)$. Now, let's take the limit of this partial sum.

$$\lim_{m\to\infty} S_m = \lim_{m\to\infty} a\left(\frac{1-r^m}{1-r}\right) = a\left(\frac{1-\lim_{m\to\infty} r^m}{1-r}\right)$$

If we want this sequence of partial sums to converge, we want to find what values of r will make $\lim_{m\to\infty} r^m$ exist.

If $r>1$ or $r\le -1$, $\lim_{m\to\infty} r^m$ does not exist.

When $|r|<1$, $\lim_{m\to\infty} r^m = 0$ and the sequence of partial sums converges.

So assuming $|r|<1$, we can find $\lim_{m\to\infty} S_m$.

$$\lim_{m\to\infty} S_m = \lim_{m\to\infty} a\left(\frac{1-r^m}{1-r}\right) = a\left(\frac{1-\lim_{m\to\infty} r^m}{1-r}\right) = a\left(\frac{1-0}{1-r}\right) = a\left(\frac{1}{1-r}\right) = \frac{a}{1-r}$$

The other possibility from above was $r=1$ and $S_m = \sum_{n=0}^{m} ar^n = a(1+m)$.

In this case, $\lim_{m\to\infty} S_m = \infty$ and therefore this geometric series would diverge.

Thus a geometric series with ratio r converges if $|r|<1$ and will have the sum

$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$.

- Notice that this characterization for the limit of the sum requires the value of the index to start at $n=0$. We will address how to calculate sums when the index starts at other values in the examples to follow.

A geometric series with ratio r diverges if $|r|\geq 1$.

Example 4: Determine whether the geometric series converges or diverges. If the series converges, find the sum.

$\sum_{n=0}^{\infty} 3\left(\frac{2}{7}\right)^n$

The ratio is $\frac{2}{7}$.

Since $\left|\frac{2}{7}\right|<1$, the series converges to $\frac{a}{1-r} = \frac{3}{1-\frac{2}{7}} = \frac{21}{5}$.

Example 5: Determine whether the geometric series converges or diverges. If the series converges, find the sum.

$$\sum_{n=1}^{\infty} \frac{\sqrt{6}}{8^n}$$

The ratio is $\frac{1}{8}$.

Since $\left|\frac{1}{8}\right| < 1$, the series converges.

Before calculating the sum, notice that the initial value of the index is $n=1$. We can calculate the sum in two different ways.

Option 1: We will use the formula for the sum of a geometric series to calculate the sum of this series, but we will need to subtract a_0 from the sum calculated by the formula.

The series converges to $\sum_{n=1}^{\infty} \frac{\sqrt{6}}{8^n} = \sum_{n=0}^{\infty} \frac{\sqrt{6}}{8^n} - a_0 = \frac{a}{1-r} - a_0 = \frac{\sqrt{6}}{1-\frac{1}{8}} - \frac{\sqrt{6}}{8^0} = \frac{8\sqrt{6}}{7} - \sqrt{6} = \frac{\sqrt{6}}{7}$.

Option 2: We can factor $\frac{1}{8}$ out of every term to adjust the series:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\sqrt{6}}{8^n} &= \frac{\sqrt{6}}{8} + \frac{\sqrt{6}}{8^2} + \frac{\sqrt{6}}{8^3} + \cdots \\ &= \frac{\sqrt{6}}{8} + \frac{\sqrt{6}}{8} \cdot \left(\frac{1}{8}\right)^1 + \frac{\sqrt{6}}{8} \cdot \left(\frac{1}{8}\right)^2 + \cdots \\ &= \sum_{n=0}^{\infty} \left(\frac{\sqrt{6}}{8} \cdot \left(\frac{1}{8}\right)^n\right).\end{aligned}$$

Now, this new expression for the series has an initial index value of $n=0$ and converges to $\frac{a}{1-r} = \frac{\frac{\sqrt{6}}{8}}{1-\frac{1}{8}} = \frac{\left(\frac{\sqrt{6}}{8}\right)}{\left(\frac{7}{8}\right)} = \frac{\sqrt{6}}{7}$.

Example 6: Determine whether the geometric series converges or diverges. If the series converges, find the sum.

$$\sum_{n=2}^{\infty}\left(-\frac{6}{11}\right)^n$$

The ratio is $-\frac{6}{11}$.

Since $\left|-\frac{6}{11}\right|<1$, the series converges.

Before calculating the sum, notice that the initial value of the index is $n=2$. We can calculate the sum in two different ways, as we did in **Example 6**.

Let's choose to use the method where we subtract a_0 and a_1 from the sum of the geometric series whose initial index is 0.

The series converges to

$$\begin{aligned}\sum_{n=2}^{\infty}\left(-\frac{6}{11}\right)^n &= \sum_{n=0}^{\infty}\left(-\frac{6}{11}\right)^n - a_0 - a_1 \\ &= \frac{a}{1-r} - a_0 - a_1 \\ &= \frac{1}{1-\left(-\frac{6}{11}\right)} - \left(-\frac{6}{11}\right)^0 - \left(-\frac{6}{11}\right)^1 \\ &= \frac{11}{17} - 1 - \left(-\frac{6}{11}\right) \\ &= \frac{36}{187}.\end{aligned}$$

Example 7: Determine whether the geometric series converges or diverges. If the series converges, find the sum.

$$\sum_{n=0}^{\infty} -\frac{1}{5}(8)^n$$

The ratio is 8. Since $|8| \geq 1$, the series diverges.

Example 8: Determine whether the geometric series converges or diverges. If the series converges, find the sum.

$$\sum_{n=1}^{\infty} \frac{e^n}{4}$$

The ratio is e. Since $|e| \geq 1$, the series diverges.

Activity Sheet with Answers

Activity: Geometric Series Name: ________________________

1. Determine if each of the following is a geometric series. If it is a geometric series, write the series in summation notation.

a) $120+40+\frac{40}{3}+\frac{40}{9}+\frac{40}{27}+\cdots$

Geometric

$$\sum_{n=0}^{\infty}120\left(\frac{1}{3}\right)^n$$

b) $-8+(-3)+2+7+12+\cdots$

Not geometric

c) $-2+4+(-8)+16+(-32)+\cdots$

Geometric

$$\sum_{n=0}^{\infty}-2(-2)^n \text{ or } \sum_{n=1}^{\infty}1(-2)^n$$

d) $\pi+\pi^2+\pi^3+\pi^4+\pi^5+\cdots$

Geometric

$$\sum_{n=0}^{\infty}\pi(\pi)^n \text{ or } \sum_{n=1}^{\infty}1(\pi)^n$$

e) $-1+7+63+215+511+\cdots$

Not geometric

f) $10+\frac{25}{2}+\frac{125}{8}+\frac{625}{32}+\frac{3125}{128}+\cdots$

Geometric

$$\sum_{n=0}^{\infty}10\left(\frac{5}{4}\right)^n$$

g) $1+4+9+16+25+\cdots$

Not geometric

h) $1000+(-250)+\frac{125}{2}+\left(-\frac{125}{8}\right)+\frac{125}{32}+\cdots$

Geometric

$$\sum_{n=0}^{\infty}1000\left(-\frac{1}{4}\right)^n$$

Determine if each of the following is a geometric series. If it is a geometric series, identify the ratio r.

2. $\sum_{n=0}^{\infty} 5\left(-\frac{2}{3}\right)^n$

Geometric, $r = -\frac{2}{3}$

3. $\sum_{n=0}^{\infty} \frac{n^3}{5}$

Not geometric

4. $\sum_{n=1}^{\infty} -3(4)^n$

Geometric, $r = 4$

5. $\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{9^n}{4}\right)$

Geometric, $r = 9$

6. $\sum_{n=3}^{\infty} \frac{7}{13^n}$

Geometric, $r = \frac{1}{13}$

7. $\sum_{n=2}^{\infty} \frac{3n^4}{6^n}$

Not geometric

8. $\sum_{n=2}^{\infty} \frac{n!}{4+3^n}$

Not geometric

9. $\sum_{n=0}^{\infty} \frac{5n-1}{8n^2-4n+2}$

Not geometric

10. $\sum_{n=1}^{\infty} \frac{1}{3}(\pi)^n$

Geometric, $r = \pi$

11. $\sum_{n=0}^{\infty} \frac{\sqrt{n^3}}{7n-9}$

Not geometric

12. A geometric series diverges if $|r| \geq 1$.

13. A geometric series converges if $|r| < 1$.

14. For a convergent geometric series, $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$.

15. Describe two ways to calculate the sum of a convergent geometric series $\sum_{n=3}^{\infty} ar^n$.

Option 1: The series converges to $\sum_{n=3}^{\infty} ar^n = \sum_{n=0}^{\infty} ar^n - a_0 - a_1 - a_2 = \frac{a}{1-r} - a_0 - a_1 - a_2$.

Option 2: We can factor ar^3 out of every term to adjust the series:

$$\begin{aligned}\sum_{n=3}^{\infty} ar^n &= ar^3 + ar^4 + ar^5 + \cdots \\ &= ar^3 + ar^3 \cdot (r)^1 + ar^3 \cdot (r)^2 + \cdots \\ &= \sum_{n=0}^{\infty} \left(ar^3 \cdot (r)^n \right).\end{aligned}$$

Now, this new expression for the series has an initial index value of $n = 0$ and converges to $\sum_{n=0}^{\infty} \left(ar^3 \cdot (r)^n \right) = \frac{ar^3}{1-r}$.

Students can verify algebraically that Option 1 and Option 2 give the same value for the series.

Determine whether each of the following geometric series converges or diverges. If the series converges, find the sum.

16. $\sum_{n=0}^{\infty} 4\left(\frac{1}{7}\right)^n$

$r = \frac{1}{7}$ and $\left|\frac{1}{7}\right| < 1$

The series converges to $\frac{4}{1-\frac{1}{7}} = \frac{14}{3}$.

17. $\sum_{n=0}^{\infty} 5\left(\frac{7}{3}\right)^n$

$r=\frac{7}{3}$ and $\left|\frac{7}{3}\right| \geq 1$

The series diverges.

18. $\sum_{n=1}^{\infty} \frac{5}{6^n}$

$r=\frac{1}{6}$ and $\left|\frac{1}{6}\right| < 1$

The series converges.

Option 1: The series converges to $\sum_{n=1}^{\infty} \frac{5}{6^n} = \frac{5}{1-\frac{1}{6}} - \frac{5}{6^0} = 1$.

Option 2: We can factor $\frac{5}{6}$ out of every term to adjust the series:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{5}{6^n} &= \frac{5}{6^1} + \frac{5}{6^2} + \frac{5}{6^3} + \cdots \\ &= \frac{5}{6} + \frac{5}{6}\left(\frac{1}{6}\right)^1 + \frac{5}{6}\left(\frac{1}{6}\right)^2 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{5}{6}\left(\frac{1}{6}\right)^n.\end{aligned}$$

Now, this new expression for the series has an initial index value of $n=0$ and converges to $\sum_{n=0}^{\infty} \frac{5}{6}\left(\frac{1}{6}\right)^n = \frac{a}{1-r} = \frac{\frac{5}{6}}{1-\frac{1}{6}} = \frac{\left(\frac{5}{6}\right)}{\left(\frac{5}{6}\right)} = 1.$

19. $\sum_{n=0}^{\infty} -\frac{3}{4}\left(\sqrt{11}\right)^n$

$r = \sqrt{11}$ and $\left|\sqrt{11}\right| \geq 1$

The series diverges.

20. $\sum_{n=2}^{\infty} -1\left(\frac{5}{6}\right)^n$

$r = \frac{5}{6}$ and $\left|\frac{5}{6}\right| < 1$

The series converges.

Option 1: The series converges to $\sum_{n=2}^{\infty} -1\left(\frac{5}{6}\right)^n = \frac{-1}{1-\frac{5}{6}} - \left(-1\left(\frac{5}{6}\right)^0\right) - \left(-1\left(\frac{5}{6}\right)^1\right) = -\frac{25}{6}$.

Option 2: We can factor $-1\left(\frac{5}{6}\right)^2$ out of every term to adjust the series:

$$\begin{aligned}
\sum_{n=2}^{\infty} -1\left(\frac{5}{6}\right)^n &= -1\left(\frac{5}{6}\right)^2 - 1\left(\frac{5}{6}\right)^3 - 1\left(\frac{5}{6}\right)^4 + \cdots \\
&= -1\left(\frac{5}{6}\right)^2 + \left(-1\left(\frac{5}{6}\right)^2\right)\left(\frac{5}{6}\right)^1 + \left(-1\left(\frac{5}{6}\right)^2\right)\left(\frac{5}{6}\right)^2 + \cdots \\
&= \sum_{n=0}^{\infty} \left(-1\left(\frac{5}{6}\right)^2\right)\left(\frac{5}{6}\right)^n .
\end{aligned}$$

Now, this new expression for the series has an initial index value of $n = 0$ and converges to $\sum_{n=0}^{\infty} \left(-1\left(\frac{5}{6}\right)^2\right)\left(\frac{5}{6}\right)^n = \frac{a}{1-r} = \frac{-1\left(\frac{5}{6}\right)^2}{1-\frac{5}{6}} = \frac{\left(-\frac{25}{36}\right)}{\left(\frac{1}{6}\right)} = -\frac{25}{6}$.

21. $\sum_{n=0}^{\infty}\left(-\frac{2}{9}\right)^n$

$r=-\frac{2}{9}$ and $\left|-\frac{2}{9}\right|<1$

The series converges to $\frac{1}{1-\left(-\frac{2}{9}\right)}=\frac{9}{11}$.

22. $\sum_{n=1}^{\infty}\frac{1}{8}\left(\sqrt{7}\right)^n$

$r=\sqrt{7}$ and $\left|\sqrt{7}\right|\geq 1$

The series diverges.

23. Write a geometric series in summation notation that converges to a sum of 6.

Answers may vary.

Sample answer: $\sum_{n=0}^{\infty}3\left(\frac{1}{2}\right)^n$

Lesson 4: *p*-Series

Teacher Notes

Overview:

In this lesson, students will learn about *p*-series. Within this lesson, students also will learn about a particular *p*-series, the harmonic series. Series that fit the *p*-series form are easy to identify and have a simple method for determining convergence. Knowing about *p*-series will be helpful when working with other convergence tests in later lessons.

Students begin by identifying *p*-series. Next, students will determine whether a given *p*-series will converge. Students will see examples where p is a whole number as well as a rational number. Additionally, students will need to recognize values of p when the terms of a series are presented in radical form.

In the coming lessons, students will begin a study of three tests that require knowledge of geometric series and *p*-series. Therefore, the summary of the activity for this lesson requires students to determine convergence of a variety of series through any method studied thus far.

Objectives:

- Identify *p*-series
- Determine the convergence of *p*-series
- Identify the harmonic series

Standards:

II. Limits: LIM-7.A.7: In addition to geometric series, common series of numbers include the harmonic series, the alternating harmonic series, and *p*-series.

Guided Notes with Answers

p-Series Name: ________________________

Example 1: Consider the series $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\cdots$.

What pattern do you notice in the terms of this series?

The denominator is the square of the term number.

What term does not seem to fit this pattern at first glance?

The first term

Can you verify that this term matches the pattern we notice?

Yes. Following the pattern, the first term is $\frac{1}{1^2}=\frac{1}{1}=1$.

Example 2: Consider the series $3+\frac{3}{8}+\frac{1}{9}+\frac{3}{64}+\frac{3}{125}+\cdots$.

What pattern do you notice in the terms of this series?

The denominator is the cube of the term number.

What terms do not seem to fit this pattern at first glance?

The first term and the third term

Can we verify that these terms match the pattern we notice?

Yes. Following the pattern, the first term is $\frac{1}{1^3}=\frac{1}{1}=1$ and the third term is $\frac{3}{3^3}=\frac{3}{27}=\frac{1}{9}$.

A series of the form $\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots$ is a *p*-series where *p* is a positive constant.

Additionally, when $p=1$, the series $\sum_{n=1}^{\infty}\frac{1}{n}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ is called the harmonic series.

The series in **Example 1** and **Example 2** are both *p*-series.

We can rewrite **Example 1** as $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n^2}$ where $p=2$.

It is important to notice that **Example 2** has a numerator other than 1. However, we know that infinite series are limits of finite sums. Based on properties of limits that we have previously studied, we know we can factor constants out of limits. Therefore, we can also factor constants out of infinite series.

$$\sum_{n=1}^{\infty}ka_n=k\sum_{n=1}^{\infty}a_n \text{ where } k \text{ is a constant.}$$

Therefore, **Example 2** can be rewritten as $3+\frac{3}{8}+\frac{1}{9}+\frac{3}{64}+\frac{3}{125}+\cdots=\sum_{n=1}^{\infty}\frac{3}{n^3}=3\sum_{n=1}^{\infty}\frac{1}{n^3}$.

Example 3: Identify the value of *p* for the *p*-series and write the series in sigma notation.

$1+\frac{1}{32}+\frac{1}{243}+\frac{1}{1024}+\frac{1}{3125}+\cdots$

$p=5$

$\sum_{n=1}^{\infty}\frac{1}{n^5}$

Example 4: Identify the value of p for the p-series and write the series in sigma notation.

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{5}}+\cdots$$

$$p=\frac{1}{2}$$

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}=\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$$

Example 5: Identify the value of p for the p-series and write the series in sigma notation.

$$7+\frac{7}{\sqrt[4]{2}}+\frac{7}{\sqrt[4]{3}}+\frac{7}{\sqrt[4]{4}}+\frac{7}{\sqrt[4]{5}}+\cdots$$

$$p=\frac{1}{4}$$

$$\sum_{n=1}^{\infty}\frac{7}{\sqrt[4]{n}}=\sum_{n=1}^{\infty}\frac{7}{n^{1/4}}=7\sum_{n=1}^{\infty}\frac{1}{n^{1/4}}$$

Example 6: Identify the value of p for the p-series and write the series in sigma notation.

$$11+\frac{11}{2\sqrt[3]{2}}+\frac{11}{3\sqrt[3]{3}}+\frac{11}{4\sqrt[3]{4}}+\frac{11}{5\sqrt[3]{5}}+\cdots$$

$$p=\frac{4}{3}$$

$$\sum_{n=1}^{\infty}\frac{11}{n\sqrt[3]{n}}=\sum_{n=1}^{\infty}\frac{11}{n^{4/3}}=11\sum_{n=1}^{\infty}\frac{1}{n^{4/3}}$$

Like finding the common ratio in a geometric series, identifying the value of p in a p-series is a crucial skill because p determines whether the series will converge or diverge.

A p-series with $p > 1$ will converge.

A p-series with $0 < p \leq 1$ will diverge.

Remember that p must be a positive constant.

Let's investigate why the convergent and divergent statements above are true.

Remember that an infinite series is a sequence of partial sums. When we determine whether $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges, we look at the sequence of partial sums.

Let's begin by looking at the partial sums of $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

$$S_1 = \frac{1}{1^p}$$

$$S_2 = \frac{1}{1^p} + \frac{1}{2^p}$$

$$S_3 = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p}$$

$$S_4 = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p}$$

and in general,

$$S_{m-1} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{(m-1)^p}$$

$$S_m = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{(m-1)^p} + \frac{1}{m^p}.$$

Think back to earlier in this course. When have we added up values of a function?

Approximating area under a curve using Riemann sums

The partial sums we have listed above look different from Riemann sums since each term is not being multiplied by Δx. However, if we let $\Delta x = 1$, we can rewrite the partial sums as follows:

$$S_1 = \frac{1}{1^p}(1)$$

$$S_2 = \frac{1}{1^p}(1) + \frac{1}{2^p}(1)$$

$$S_3 = \frac{1}{1^p}(1) + \frac{1}{2^p}(1) + \frac{1}{3^p}(1)$$

$$S_4 = \frac{1}{1^p}(1) + \frac{1}{2^p}(1) + \frac{1}{3^p}(1) + \frac{1}{4^p}(1)$$

and in general,

$$S_{m-1} = \frac{1}{1^p}(1) + \frac{1}{2^p}(1) + \frac{1}{3^p}(1) + \frac{1}{4^p}(1) + \cdots + \frac{1}{(m-1)^p}(1)$$

$$S_m = \frac{1}{1^p}(1) + \frac{1}{2^p}(1) + \frac{1}{3^p}(1) + \frac{1}{4^p}(1) + \cdots + \frac{1}{(m-1)^p}(1) + \frac{1}{m^p}(1).$$

Now, we can see that each of these partial sums can be interpreted as a Riemann sum.

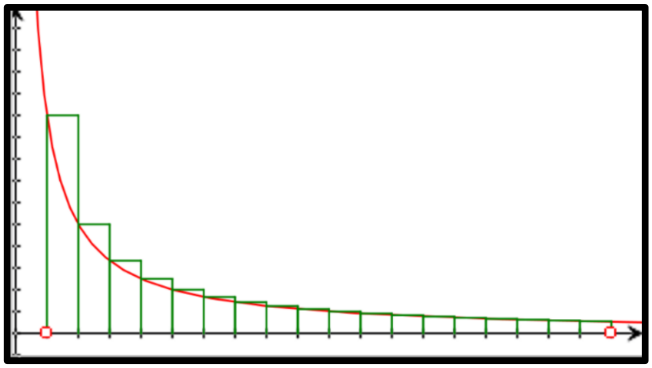
Let's look at Riemann sum approximations for $\int_1^m \frac{1}{x^p}\,dx$.

A left Riemann sum approximation with $\Delta x = 1$ would be:

$$\int_1^m \frac{1}{x^p}\,dx \approx \frac{1}{1^p}(1) + \frac{1}{2^p}(1) + \frac{1}{3^p}(1) + \frac{1}{4^p}(1) + \ldots + \frac{1}{(m-1)^p}(1).$$

This would correspond to the partial sum S_{m-1}.

We know that $p>0$ and thus $f(x)=\frac{1}{x^p}$ is a decreasing function on the interval $(0,\infty)$.



Left Riemann sum approximation [4].

Therefore, the left Riemann sum approximation is an overestimate of the integral.

That is, $S_{m-1} \geq \int_1^m \frac{1}{x^p}\,dx$.

For $0<p<1$,

$$\begin{aligned}\lim_{m\to\infty} S_{m-1} &\geq \lim_{m\to\infty}\int_1^m \frac{1}{x^p}\,dx\\ &= \lim_{m\to\infty} \left.\frac{x^{(-p+1)}}{-p+1}\right|_1^m\\ &= \lim_{m\to\infty}\left(\frac{m^{(-p+1)}}{-p+1}-\frac{1^{(-p+1)}}{-p+1}\right).\end{aligned}$$

When $0<p<1$, $-p+1$ is positive and therefore $m^{(-p+1)}$ grows without bound as $m\to\infty$. Thus $\lim_{m\to\infty}\left(\frac{m^{(-p+1)}}{-p+1}-\frac{1^{(-p+1)}}{-p+1}\right)$ is infinite. This implies that $\lim_{m\to\infty} S_{m-1}$ is infinite and the sequence of partial sums diverges. Thus, $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges.

For $p = 1$,

$$\begin{aligned}\lim_{m\to\infty} S_{m-1} &\geq \lim_{m\to\infty} \int_1^m \frac{1}{x^1}\,dx \\ &= \lim_{m\to\infty} \ln|x|\Big|_1^m \\ &= \lim_{m\to\infty} \left(\ln|m| - \ln|1|\right) \\ &= \lim_{m\to\infty} \left(\ln|m| - 0\right) \\ &= \infty.\end{aligned}$$

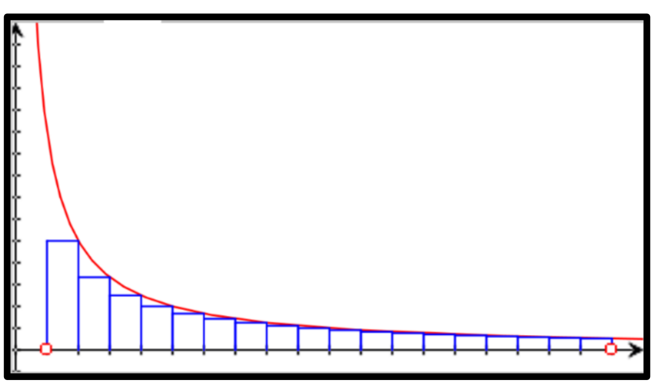
Thus $\lim_{m\to\infty} S_{m-1}$ is infinite and therefore the sequence of partial sums diverges. Thus, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

A right Riemann sum approximation with $\Delta x = 1$ would be:

$$\int_1^m \frac{1}{x^p}\,dx \approx \frac{1}{2^p}(1) + \frac{1}{3^p}(1) + \frac{1}{4^p}(1) + \cdots + \frac{1}{(m-1)^p}(1) + \frac{1}{m^p}(1).$$

This would correspond to the partial sum $S_m - \frac{1}{1^p}(1)$.

We know that $p > 0$ and thus $f(x) = \frac{1}{x^p}$ is a decreasing function on the interval $(0, \infty)$.



Right Riemann sum approximation [4].

Therefore, the right Riemann sum approximation is an underestimate of the integral.

That is, $S_m - \frac{1}{1^p}(1) \le \int_1^m \frac{1}{x^p}\,dx$, so $S_m \le \frac{1}{1^p}(1) + \int_1^m \frac{1}{x^p}\,dx$.

For $p > 1$,

$$\begin{aligned}\lim_{m\to\infty} S_m &\le \lim_{m\to\infty}\left(\frac{1}{1^p}(1) + \int_1^m \frac{1}{x^p}\,dx\right)\\ &= \frac{1}{1^p}(1) + \lim_{m\to\infty}\int_1^m \frac{1}{x^p}\,dx\\ &= \frac{1}{1^p}(1) + \lim_{m\to\infty}\left.\frac{x^{(-p+1)}}{-p+1}\right|_1^m\\ &= \frac{1}{1^p}(1) + \lim_{m\to\infty}\left(\frac{m^{(-p+1)}}{-p+1} - \frac{1^{(-p+1)}}{-p+1}\right).\end{aligned}$$

Thus, $\lim_{m\to\infty} S_m \le \frac{1}{1^p}(1) + \lim_{m\to\infty}\left(\frac{m^{(-p+1)}}{-p+1} - \frac{1^{(-p+1)}}{-p+1}\right)$.

When $p > 1$, $-p+1$ is negative and therefore $m^{(-p+1)}$ converges to 0 as $m \to \infty$. Thus $\frac{1}{1^p}(1) + \lim_{m\to\infty}\left(\frac{m^{(-p+1)}}{-p+1} - \frac{1^{(-p+1)}}{-p+1}\right)$ exists. This implies that $\lim_{m\to\infty} S_m$ converges. Since the limit of partial sums converges, $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges.

Therefore, $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges when $p > 1$ and diverges when $0 < p \le 1$.

We will usually assign *p*-series an initial index value of $n = 1$, however the initial value of the index can be any finite positive value. This will not affect the convergence of a *p*-series.

Using this information, we can determine whether the series in **Examples 3** through **6** converge.

Example 3: $p = 5$ and $5 > 1$ The series converges.

Example 4: $p = \frac{1}{2}$ and $0 < \frac{1}{2} \leq 1$ The series diverges.

Example 5: $p = \frac{1}{4}$ and $0 < \frac{1}{4} \leq 1$ The series diverges.

Example 6: $p = \frac{4}{3}$ and $\frac{4}{3} > 1$ The series converges.

Example 7: Determine whether the *p*-series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n^{6/7}} = 1 + \frac{1}{2^{6/7}} + \frac{1}{3^{6/7}} + \frac{1}{4^{6/7}} + \frac{1}{5^{6/7}} + \cdots$$

$p = \frac{6}{7}$ and $0 < \frac{6}{7} \leq 1$

The series diverges.

Example 8: Determine whether the *p*-series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{9}{n^{\pi}} = 9 + \frac{9}{2^{\pi}} + \frac{9}{3^{\pi}} + \frac{9}{4^{\pi}} + \frac{9}{5^{\pi}} + \cdots$$

$p = \pi$ and $\pi > 1$

The series converges.

Example 9: Determine whether the *p*-series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{3}{\sqrt[6]{n}} = 3 + \frac{3}{\sqrt[6]{2}} + \frac{3}{\sqrt[6]{3}} + \frac{3}{\sqrt[6]{4}} + \frac{3}{\sqrt[6]{5}} + \cdots$$

$p = \frac{1}{6}$ and $0 < \frac{1}{6} \leq 1$

The series diverges.

Example 10: Determine whether the *p*-series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{8.3}{n} = 8.3 + \frac{8.3}{2} + \frac{8.3}{3} + \frac{8.3}{4} + \frac{8.3}{5} + \cdots$$

$p = 1$ and $0 < 1 \leq 1$

The series diverges. This is a multiple of the harmonic series.

Activity Sheet with Answers

Activity: p-Series

Name: ________________________

1. Determine if each of the following is a *p*-series. If it is, write the *p*-series in summation notation.

a) $1+\frac{1}{2^{-1}}+\frac{1}{3^{-1}}+\frac{1}{4^{-1}}+\frac{1}{5^{-1}}+\ldots$

Not a *p*-series.

b) $7+\frac{7}{2}+\frac{7}{3}+\frac{7}{4}+\frac{7}{5}+\cdots$

p-series with $p=1$; multiple of the harmonic series

$$\sum_{n=1}^{\infty}\frac{7}{n}$$

c) $17+\frac{17}{2\sqrt[9]{2}}+\frac{17}{3\sqrt[9]{3}}+\frac{17}{4\sqrt[9]{4}}+\frac{17}{5\sqrt[9]{5}}+\ldots$

p-series with $p=\frac{10}{9}$

$$\sum_{n=1}^{\infty}\frac{17}{n\sqrt[9]{n}}=\sum_{n=1}^{\infty}\frac{17}{n^{10/9}}$$

d) $7+7\left(\frac{5}{2}\right)^{1}+7\left(\frac{5}{2}\right)^{2}+7\left(\frac{5}{2}\right)^{3}+7\left(\frac{5}{2}\right)^{4}+\cdots$

Not a *p*-series.

e) $1^{-5}+2^{-5}+3^{-5}+4^{-5}+5^{-5}+\cdots$

p-series with $p=5$

$$\sum_{n=1}^{\infty}n^{-5}=\sum_{n=1}^{\infty}\frac{1}{n^{5}}$$

f) $4+\frac{1}{2}+\frac{4}{27}+\frac{1}{16}+\frac{4}{125}+\cdots$

p-series with $p=3$

$$\sum_{n=1}^{\infty}\frac{4}{n^{3}}$$

Identify p in each of the following p-series. Then, determine if the series converges or diverges.

2. $\sum_{n=1}^{\infty} \frac{1}{n^{3/7}}$

$p = \frac{3}{7}$ and $0 < \frac{3}{7} \leq 1$

The series diverges.

3. $\sum_{n=1}^{\infty} \frac{1}{n^{e}}$

$p = e$ and $e > 1$

The series converges.

4. Write a p-series in summation notation with $p = \frac{1}{45}$ and whose first term is 9.

$$\sum_{n=1}^{\infty} \frac{9}{n^{1/45}}$$

5. Write a p-series in summation notation with $p = 6$ and whose first term is 1.

$$\sum_{n=1}^{\infty} \frac{1}{n^{6}}$$

Determine whether each of the following series converges or diverges. Use any method we have studied.

6. $\sum_{n=1}^{\infty} \frac{8+3n}{9n}$

Use the n^{th}-Term Test for Divergence.

$a_n = \frac{8+3n}{9n}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{8+3n}{9n} = \frac{1}{3}$

Therefore, $\lim_{n\to\infty} a_n \neq 0$ and the series diverges.

7. $\sum_{n=2}^{\infty} 5(3)^n$

Geometric series

$r = 3$ and $|3| \geq 1$

The series diverges.

8. $\sum_{n=1}^{\infty} \frac{13}{n^2 \sqrt[5]{n}}$

p-Series

$p = \frac{11}{5}$ and $\frac{11}{5} > 1$

The series converges.

9. $\sum_{n=0}^{\infty} \frac{-2^n}{3}$

Geometric series

$r = 2$ and $|2| \geq 1$

The series diverges.

10. $\sum_{n=2}^{\infty} \frac{5.2^n}{n^3}$

Use the n^{th}-Term Test for Divergence.

$a_n = \frac{5.2^n}{n^3}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5.2^n}{n^3} = \infty$

Therefore, $\lim_{n\to\infty} a_n \neq 0$ and the series diverges.

11. $\sum_{n=1}^{\infty} \frac{-3}{n^8}$

p-Series

$p = 8$ and $8 > 1$

The series converges.

12. $\sum_{n=0}^{\infty} \pi \left(\frac{1}{6}\right)^n$

Geometric series

$r = \frac{1}{6}$ and $\left|\frac{1}{6}\right| < 1$

The series converges.

13. $\sum_{n=1}^{\infty} \frac{4}{\sqrt{n}}$

p-Series

$p = \frac{1}{2}$ and $0 < \frac{1}{2} \leq 1$

The series diverges.

14. $\sum_{n=3}^{\infty} 16.2$

Use the n^{th}-Term Test for Divergence.

$a_n = 16.2$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} 16.2 = 16.2$

Therefore, $\lim_{n\to\infty} a_n \neq 0$ and the series diverges.

15. $\sum_{n=0}^{\infty} \frac{4^n}{3\cdot 5^n}$

Geometric series

$r = \frac{4}{5}$ and $\left|\frac{4}{5}\right| < 1$

The series converges.

16. Complete the following chart.

	General form of series	Series diverges if …	Series converges if…
The n^{th}-Term Test for Divergence	$\sum_{n=k}^{\infty} a_n$	$\lim_{n\to\infty} a_n \neq 0$	Not applicable
Geometric Series	$\sum_{n=k}^{\infty} ar^n$	$\lvert r\rvert \geq 1$	$\lvert r\rvert < 1$
p-Series	$\sum_{n=m}^{\infty} \frac{1}{n^p}$	$0 < p \leq 1$	$p > 1$

Note: Let k represent any integer and let m represent any positive integer.

17. The n^{th}-Term Test for Divergence can only determine that a series diverges.

18. We can calculate the sum of a convergent geometric series with an initial index value of $n = 0$ using the formula $\frac{a}{1-r}$.

19. In a p-Series in the form $\sum_{n=m}^{\infty} \frac{1}{n^p}$, p must be a positive constant.

Lesson 5: Direct Comparison Test

Teacher Notes

Overview:

In this lesson, students will learn about the Direct Comparison Test. To apply this test, students must use prior knowledge of geometric series and *p*-series. Students should be familiar with characteristics of each type of series and be able to identify when these series converge or diverge. Additionally, when given two series, students will need to determine which series has larger or smaller terms.

In the activity, students will identify a series that is similar to a given series. Next, students will describe scenarios that are supported by the Direct Comparison Test. I have also included a question asking students to describe the two cases when the Direct Comparison Test is inconclusive. Additionally, students will need to apply the Direct Comparison Test to a given series.

As the final activity, I have asked students to decide what test or method would be best to use to determine if a given series converges or diverges.

Objectives:

- Identify series similar to a provided series
- Apply convergence rules of geometric series and *p*-series
- Given two series, justify which series has smaller terms
- Given two series, justify which series has larger terms
- Apply the Direct Comparison Test

Standards:

II. Limits: LIM-7.A.8: The comparison test is a method to determine whether a series converges or diverges.

Guided Notes with Answers

Direct Comparison Test Name: ________________________

Example 1: We are given the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ and asked to determine whether the series converges or diverges. What methods might we try in order to answer this question?

1. The n^{th}-Term Test for Divergence

   $a_n = \frac{1}{n^2+1}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2+1} = 0$

   Since $\lim_{n\to\infty} a_n = 0$, we cannot make a conclusion about whether $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ diverges. The n^{th}-Term Test is inconclusive for this series.

2. Geometric Series

   The series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is not of the form $\sum_{n=0}^{\infty} ar^n$. Additionally, we cannot factor out a value to adjust the given series to fit the form. This is not a geometric series. We cannot use what we know about geometric series to determine whether the series converges or diverges.

3. *p*-Series

   The series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ does not fit the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$. This is not a *p*-series. We cannot use what we know about *p*-series to determine whether the series converges or diverges.

None of the methods we have studied so far is applicable to this particular series.

What if instead of matching a known series type exactly, we looked for a type that was *similar* to the series we are working with?

The series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is *similar* to a *p*-series.

What *p*-series would we say is most *similar* to $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$?

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

What do we know about the series we wrote above?

In the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, $p=2$ and $2>1$. Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Let's list a few terms of the original series we were given and the new series we have compared it to.

Original: $\sum_{n=1}^{\infty} \frac{1}{n^2+1} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \cdots$

p-Series: $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$

Let's compare the two series term by term.

$\frac{1}{2} < 1$ $\quad\quad$ $\frac{1}{5} < \frac{1}{4}$ $\quad\quad$ $\frac{1}{10} < \frac{1}{9}$ $\quad\quad$ $\frac{1}{17} < \frac{1}{16}$

We can see that each term of the original series is less than the corresponding term of the series we selected.

Additionally, we can compare the expression for the n^{th} term for each series.

$a_n = \frac{1}{n^2+1}$ and $b_n = \frac{1}{n^2}$

What do we notice about these expressions?

The numerator is the same for each expression. However, a_n has a larger denominator and will result in smaller terms.

Since we know that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, what do you think this means about our original series?

Since the terms of the original series are positive and smaller than the terms of the *p*-series, the partial sums of the original series must be positive and smaller than the partial sums of the *p*-series. We know that the sequence of partial sums of the *p*-series converges, and thus the *p*-series converges.

Therefore, it makes sense that the sequence of partial sums of the original series must converge as well, resulting in the original series converging.

Although this is not a formal proof, there is a proof which you may learn in a college calculus course.

This leads us to the next convergence test we will study: the Direct Comparison Test.

Direct Comparison Test

Given two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, let $0 < a_n \le b_n$ for all *n*.

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

To use this test, we must find a series similar to the series we are given. We should choose a series that is either a geometric series or a *p*-series since we can easily determine whether these series converge or diverge.

Additionally, notice that this test requires both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ to consist of positive terms. This is an important condition to verify before applying the Direct Comparison Test.

Let's investigate why this condition is important. We will assume we only need $a_n \le b_n$.

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{-1}{n}$ and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$.

It is true that $a_n \le b_n$ for all terms since every a_n is negative and every b_n is positive. We know that $\sum_{n=1}^{\infty} \frac{-1}{n}$ diverges since it is the negative of the harmonic series and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a *p*-series that converges. So even with $a_n \le b_n$, when the terms of the series are not all positive, we cannot use the convergence of $\sum_{n=1}^{\infty} b_n$ to determine the convergence of $\sum_{n=1}^{\infty} a_n$. Also even with $a_n \le b_n$, we cannot use divergence of $\sum_{n=1}^{\infty} a_n$ to determine the divergence of $\sum_{n=1}^{\infty} b_n$.

Therefore, we must be sure that we are working with series consisting of positive terms when we apply this test.

Example 2: Use the Direct Comparison Test to determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{5}{\sqrt[3]{n}}$$

We will compare this series to the p-series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$.

Note that both the original series $\sum_{n=1}^{\infty} \frac{5}{\sqrt[3]{n}}$ and new series $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ consist of positive terms.

In the series $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$, $p = \frac{1}{3}$ and $0 < \frac{1}{3} \le 1$. Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ diverges.

Let $a_n = \frac{1}{\sqrt[3]{n}}$ and $b_n = \frac{5}{\sqrt[3]{n}}$.

Notice that a_n and b_n have the same denominator. However, b_n has a larger numerator and therefore $a_n < b_n$. Since we know that $\sum_{n=1}^{\infty} a_n$ diverges, we know that $\sum_{n=1}^{\infty} b_n$ diverges as well by the Direct Comparison Test.

Thus, the series $\sum_{n=1}^{\infty} \frac{5}{\sqrt[3]{n}}$ diverges.

We could have also determined that the original series $\sum_{n=1}^{\infty} \frac{5}{\sqrt[3]{n}}$ diverges by noticing that it is a multiple of the p-series we used in the comparison.

Example 3: Use the Direct Comparison Test to determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{2^n}{3^n + 5}$$

We will compare this series to the geometric series $\sum_{n=1}^{\infty} \frac{2^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$.

Note that both the original series $\sum_{n=1}^{\infty} \frac{2^n}{3^n + 5}$ and new series $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ consist of positive terms.

In the series $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$, $r = \frac{2}{3}$ and $\left|\frac{2}{3}\right| < 1$. Therefore, $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ converges.

Let $a_n = \frac{2^n}{3^n + 5}$ and $b_n = \frac{2^n}{3^n}$.

Notice that a_n and b_n have the same numerator. However, a_n has a larger denominator and therefore $a_n < b_n$. Since we know that $\sum_{n=1}^{\infty} b_n$ converges, we know that $\sum_{n=1}^{\infty} a_n$ converges as well by the Direct Comparison Test.

Thus, the series $\sum_{n=1}^{\infty} \frac{2^n}{3^n + 5}$ converges.

Example 4: Use the Direct Comparison Test to determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{8}{n+3}$$

We will compare this series to the harmonic series $\sum_{n=1}^{\infty} \frac{8}{n}$.

Note that both the original series $\sum_{n=1}^{\infty} \frac{8}{n+3}$ and new series $\sum_{n=1}^{\infty} \frac{8}{n}$ consist of positive terms.

In the series $\sum_{n=1}^{\infty} \frac{8}{n}$, $p=1$ and $0<1\le 1$. Therefore, $\sum_{n=1}^{\infty} \frac{8}{n}$ diverges.

Let $a_n = \frac{8}{n+3}$ and $b_n = \frac{8}{n}$.

Notice that a_n and b_n have the same numerator. However, a_n has a larger denominator and therefore $a_n < b_n$.

We know that $\sum_{n=1}^{\infty} b_n$ diverges, but we cannot make a conclusion about $\sum_{n=1}^{\infty} a_n$ using the Direct Comparison Test.

We will investigate another method to determine the convergence or divergence of a series in the next lesson that will address this situation.

Example 5: Use the Direct Comparison Test to determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^5+1}}$$

We will compare this series to the *p*-series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^5}} = \sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$.

Note that both the original series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^5+1}}$ and new series $\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ consist of positive terms.

In the series $\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$, $p = \frac{5}{2}$ and $\frac{5}{2} > 1$. Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ converges.

Let $a_n = \frac{1}{\sqrt{n^5+1}}$ and $b_n = \frac{1}{\sqrt{n^5}}$.

Notice that a_n and b_n have the same numerator. However, a_n has a larger denominator and therefore $a_n < b_n$. Since we know that $\sum_{n=1}^{\infty} b_n$ converges, we know that $\sum_{n=1}^{\infty} a_n$ converges as well by the Direct Comparison Test.

Thus, the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^5+1}}$ converges.

Example 6: Use the Direct Comparison Test to determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{7\sqrt[4]{n}-2}$$

We will compare this series to a multiple of a *p*-series with $p = \frac{1}{4}$. We will use the series $\sum_{n=1}^{\infty} \frac{1}{7\sqrt[4]{n}} = \sum_{n=1}^{\infty} \frac{1}{7n^{1/4}}$.

Note that both the original series $\sum_{n=1}^{\infty} \frac{1}{7\sqrt[4]{n}-2}$ and new series $\sum_{n=1}^{\infty} \frac{1}{7n^{1/4}}$ consist of positive terms.

In the series $\sum_{n=1}^{\infty} \frac{1}{7n^{1/4}}$, $p = \frac{1}{4}$ and $0 < \frac{1}{4} \leq 1$. Therefore, $\sum_{n=1}^{\infty} \frac{1}{7n^{1/4}}$ diverges.

Let $a_n = \frac{1}{7\sqrt[4]{n}}$ and $b_n = \frac{1}{7\sqrt[4]{n}-2}$.

Notice that a_n and b_n have the same numerator. However, b_n has a smaller denominator and therefore $a_n < b_n$. Since we know that $\sum_{n=1}^{\infty} a_n$ diverges, we know that $\sum_{n=1}^{\infty} b_n$ diverges as well by the Direct Comparison Test.

Thus, the series $\sum_{n=1}^{\infty} \frac{1}{7\sqrt[4]{n}-2}$ diverges.

Activity Sheet with Answers

Activity: Direct Comparison Test Name: ______________________

Determine if each of the following is similar to a geometric series or a *p*-series. Then provide the series you would use with the Direct Comparison Test.

1. $\sum_{n=1}^{\infty} \frac{5^n + 2}{4^n}$

Geometric series

$$\sum_{n=1}^{\infty} \left(\frac{5}{4}\right)^n$$

2. $\sum_{n=1}^{\infty} \frac{1}{3n^3 + 1}$

p-series

$$\sum_{n=1}^{\infty} \frac{1}{3n^3} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n^3}$$

3. $\sum_{n=1}^{\infty} \frac{1}{10\sqrt[7]{n^5} - 9}$

p-series

$$\sum_{n=1}^{\infty} \frac{1}{10\sqrt[7]{n^5}} = \frac{1}{10}\sum_{n=1}^{\infty} \frac{1}{\sqrt[7]{n^5}}$$

4. $\sum_{n=1}^{\infty} \frac{6^n}{11^n + 7}$

Geometric series

$$\sum_{n=1}^{\infty} \left(\frac{6}{11}\right)^n$$

Use the words “smaller” and “larger” to complete the following statements describing the results of the Direct Comparison Test.

5. If the larger series converges, the smaller series must also converge.

6. If the smaller series diverges, the larger series must also diverge.

Determine whether each of the following series converges or diverges using the Direct Comparison Test.

7. $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n^6+12}}$

We will compare this series to the *p*-series $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n^6}} = \sum_{n=1}^{\infty} \frac{2}{n^3}$.

Note that both the original series $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n^6+12}}$ and new series $\sum_{n=1}^{\infty} \frac{2}{n^3}$ consist of positive terms.

In the series $\sum_{n=1}^{\infty} \frac{2}{n^3}$, $p=3$ and $3>1$. Therefore, $\sum_{n=1}^{\infty} \frac{2}{n^3}$ converges.

Let $a_n = \frac{2}{\sqrt{n^6+12}}$ and $b_n = \frac{2}{\sqrt{n^6}}$.

Notice that a_n and b_n have the same numerator. However, a_n has a larger denominator and therefore $a_n < b_n$. Since we know that $\sum_{n=1}^{\infty} b_n$ converges, we know that $\sum_{n=1}^{\infty} a_n$ converges as well by the Direct Comparison Test.

Thus, the series $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n^6+12}}$ converges.

8. $\sum_{n=1}^{\infty} \frac{8^n}{7^n - 4}$

We will compare this series to the geometric series $\sum_{n=1}^{\infty} \frac{8^n}{7^n} = \sum_{n=1}^{\infty} \left(\frac{8}{7}\right)^n$.

Note that both the original series $\sum_{n=1}^{\infty} \frac{8^n}{7^n - 4}$ and new series $\sum_{n=1}^{\infty} \left(\frac{8}{7}\right)^n$ consist of positive terms.

In the series $\sum_{n=1}^{\infty} \left(\frac{8}{7}\right)^n$, $r = \frac{8}{7}$ and $\left|\frac{8}{7}\right| \geq 1$. Therefore, $\sum_{n=1}^{\infty} \left(\frac{8}{7}\right)^n$ diverges.

Let $a_n = \frac{8^n}{7^n}$ and $b_n = \frac{8^n}{7^n - 4}$.

Notice that a_n and b_n have the same numerator. However, b_n has a smaller denominator and therefore $a_n < b_n$. Since we know that $\sum_{n=1}^{\infty} a_n$ diverges, we know that $\sum_{n=1}^{\infty} b_n$ diverges as well by the Direct Comparison Test.

Thus, the series $\sum_{n=1}^{\infty} \frac{8^n}{7^n - 4}$ diverges.

9. Given two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ with $0 < a_n \le b_n$, describe two scenarios that are inconclusive when using the Direct Comparison Test.

Scenario 1: We can prove that $\sum_{n=1}^{\infty} a_n$ converges. Then we cannot make a conclusion about $\sum_{n=1}^{\infty} b_n$ using the Direct Comparison Test.

Scenario 2: We can prove that $\sum_{n=1}^{\infty} b_n$ diverges. Then we cannot make a conclusion about $\sum_{n=1}^{\infty} a_n$ using the Direct Comparison Test.

State the method you would use to determine the convergence of each of the following series. You do not need to determine whether the series converges or diverges.

10. $\sum_{n=1}^{\infty} \frac{6}{\sqrt[5]{n^3}}$

p-series

11. $\sum_{n=1}^{\infty} \frac{9}{5^n}$

Geometric series

12. $\sum_{n=1}^{\infty} \frac{2^n}{3+7^n}$

Direct Comparison Test

13. $\sum_{n=1}^{\infty} \frac{11n-8}{5n+7}$

The n^{th}-Term Test for Divergence

Lesson 6: Limit Comparison Test

<u>Teacher Notes</u>

Overview:

In this lesson, students will learn about the Limit Comparison Test. Similar to the Direct Comparison Test, students will need to use prior knowledge of geometric series and *p*-series. When given a series that does not fit the form of a geometric series or a *p*-series, students will identify a similar series that is one of these types of series. Students will then apply the Limit Comparison Test to make a conclusion about the original series.

This test is important since it often can be used in place of the Direct Comparison Test, and can be applied in more situations. Unlike the Direct Comparison Test, students do not need to determine what series produces larger or smaller terms. Instead, students must take a limit. Students are very familiar with limits since the beginning of the calculus course focuses on computing limits through various methods.

The activity for this lesson is very straightforward. Students are asked to apply the Limit Comparison Test to a variety of series. Additionally, students are asked to identify when they would attempt to use this test. Finally, students will describe what limits will allow them to apply the Limit Comparison Test.

Objectives:

- Identify series similar to a provided series
- Apply convergence rules of geometric series and *p*-series
- Find the limit of a ratio
- Apply the Limit Comparison Test

Standards:

II. Limits: LIM-7.A.9: The limit comparison test is a method to determine whether a series converges or diverges.

Guided Notes with Answers

Limit Comparison Test Name: ________________________

Example 1: We are given the series $\sum_{n=1}^{\infty} \frac{7}{6^n - 5}$ and asked to determine whether the series converges or diverges. What methods might we try in order to answer this question?

Note: Our instincts should lead us to the Direct Comparison Test, but let's investigate the other methods we have learned as well.

1. The n^{th}-Term Test for Divergence

   $a_n = \frac{7}{6^n - 5}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{7}{6^n - 5} = 0$

   Since $\lim_{n\to\infty} a_n = 0$, we cannot make a conclusion about whether $\sum_{n=1}^{\infty} \frac{7}{6^n - 5}$ diverges. The n^{th}-Term Test is inconclusive for this series.

2. Geometric Series

   The series $\sum_{n=1}^{\infty} \frac{7}{6^n - 5}$ is not of the form $\sum_{n=0}^{\infty} ar^n$. Additionally, we cannot factor out a value to adjust the given series to fit the form. This is not a geometric series. We cannot use what we know about geometric series to determine whether the series converges or diverges.

3. *p*-Series

   The series $\sum_{n=1}^{\infty} \frac{7}{6^n - 5}$ does not fit the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$. This is not a *p*-series. We cannot use what we know about *p*-series to determine whether the series converges or diverges.

4. Direct Comparison Test

   The series $\sum_{n=1}^{\infty} \frac{7}{6^n - 5}$ is similar to the geometric series $\sum_{n=1}^{\infty} \frac{7}{6^n} = \sum_{n=1}^{\infty} 7\left(\frac{1}{6}\right)^n$.

Note that both the original series $\sum_{n=1}^{\infty} \frac{7}{6^n - 5}$ and new series $\sum_{n=1}^{\infty} 7\left(\frac{1}{6}\right)^n$ consist of positive terms.

In the series $\sum_{n=1}^{\infty} 7\left(\frac{1}{6}\right)^n$, $r = \frac{1}{6}$ and $\left|\frac{1}{6}\right| < 1$. Therefore, $\sum_{n=1}^{\infty} 7\left(\frac{1}{6}\right)^n$ converges.

Let $a_n = \frac{7}{6^n - 5}$ and $b_n = \frac{7}{6^n}$.

Notice that a_n and b_n have the same numerator. However, a_n has a smaller denominator and therefore $a_n > b_n$.

Notice that this is a different scenario than we need in order to apply the Direct Comparison Test.

Although we know that $\sum_{n=1}^{\infty} b_n$ converges, we cannot make a conclusion about $\sum_{n=1}^{\infty} a_n$ using the Direct Comparison Test.

None of the methods we have studied so far is applicable to this particular series.

However, there is another test we can use to determine the convergence or divergence of a series: The Limit Comparison Test.

As in the Direct Comparison Test, the Limit Comparison Test will require us to use a series that is *similar* to the series we are given.

Limit Comparison Test

Suppose that $a_n > 0$, $b_n > 0$, and $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L$ where *L* is finite and positive. Then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Let's take a look at the possibilities for *L* to see why *L* must be finite and positive in order to apply the Limit Comparison Test.

Since we must begin with $a_n > 0$ and $b_n > 0$, we know that the limit of the ratio of these terms must be nonnegative if the limit exists. Therefore, we know $L \geq 0$.

Case 1: If *L* is a positive finite value, we can conclude that the limit of the ratio of terms of $\frac{a_n}{b_n}$ is this same positive finite value. Therefore when we look far enough out in the series, the terms of $\sum a_n$ are about *L* times the size of the terms of $\sum b_n$.

So the partial sums for $\sum a_n$ converge if and only if the partial sums for $\sum b_n$ converge. Likewise, the partial sums for $\sum a_n$ diverge if and only if the partial sums for $\sum b_n$ diverge.

Case 2: If $L = 0$ or the limit does not exist, the Limit Comparison Test is inconclusive. The following examples demonstrates this.

Let's use the series $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ and $\sum_{n=1}^{\infty}\frac{1}{n}$. Both series consist of positive terms.

Let $a_n = \frac{1}{n^2+1}$ and $b_n = \frac{1}{n}$.

Now, find $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)$.

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)=\lim_{n\to\infty}\frac{1}{n^2+1}\div\frac{1}{n}=\lim_{n\to\infty}\frac{1}{n^2+1}\cdot\frac{n}{1}=\lim_{n\to\infty}\frac{n}{n^2+1}=0$$

Let's change the labels and find $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)$ once again.

Let $a_n=\frac{1}{n}$ and $b_n=\frac{1}{n^2+1}$.

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)=\lim_{n\to\infty}\frac{1}{n}\div\frac{1}{n^2+1}=\lim_{n\to\infty}\frac{1}{n}\cdot\frac{n^2+1}{1}=\lim_{n\to\infty}\frac{n^2+1}{n}=\infty$$

We know $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ converges by the Direct Comparison Test (see Lesson 5) and we know that $\sum_{n=1}^{\infty}\frac{1}{n}$ is the harmonic series which diverges. So in these examples, when $L=0$ or the limit does not exist, one series converges while the other diverges.

Note: There are other examples where both series converge or both series diverge even when L is not finite and positive.

To summarize, we have shown that we can only apply the Limit Comparison Test to two series consisting of positive terms if the limit of the ratio of their terms is a finite, positive value.

Going back to **Example 1**, we can compare $\sum_{n=1}^{\infty}\frac{7}{6^n-5}$ to $\sum_{n=1}^{\infty}\left(\frac{1}{6}\right)^n$. When we were trying to apply the Direct Comparison Test, we used $\sum_{n=1}^{\infty}7\left(\frac{1}{6}\right)^n$ as the comparison series but were unable to apply the test because $\sum_{n=1}^{\infty}7\left(\frac{1}{6}\right)^n$ has smaller terms than $\sum_{n=1}^{\infty}\frac{7}{6^n-5}$.

Since we do not need to decide what series has larger or smaller terms in order to use the Limit Comparison Test, we will use the simpler version of the geometric series in the comparison.

The series $\sum_{n=1}^{\infty}\left(\frac{1}{6}\right)^n$ is a geometric series with $r=\frac{1}{6}$ and $\left|\frac{1}{6}\right|<1$. Therefore, $\sum_{n=1}^{\infty}\left(\frac{1}{6}\right)^n$ converges.

Note that both the original series $\sum_{n=1}^{\infty}\frac{7}{6^n-5}$ and new series $\sum_{n=1}^{\infty}\left(\frac{1}{6}\right)^n$ consist of positive terms.

Let's apply the Limit Comparison Test.

Let $a_n=\frac{7}{6^n-5}$ and $b_n=\left(\frac{1}{6}\right)^n$.

Now, find $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)$.

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)=\lim_{n\to\infty}\frac{\frac{7}{6^n-5}}{\left(\frac{1}{6}\right)^n}=\lim_{n\to\infty}\frac{7}{6^n-5}\cdot\frac{6^n}{1}=\lim_{n\to\infty}\frac{7\left(6^n\right)}{6^n-5}=7$$

Since $L=7$ and 7 is finite and positive, both series converge.

What if we interchanged the labels before calculating the limit?

Let $a_n = \left(\frac{1}{6}\right)^n$ and $b_n = \frac{7}{6^n - 5}$.

Find $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)$.

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\frac{\left(\frac{1}{6}\right)^n}{\frac{7}{6^n - 5}} = \lim_{n\to\infty}\frac{1}{6^n}\cdot\frac{6^n - 5}{7} = \lim_{n\to\infty}\frac{6^n - 5}{7\left(6^n\right)} = \frac{1}{7}$$

Since $L = \frac{1}{7}$ and $\frac{1}{7}$ is finite and positive, both series converge.

Notice that it did not matter which series we labeled as $\sum a_n$ or $\sum b_n$. Both limits produced positive, finite values.

When using the Limit Comparison Test, we can choose to label the given series as either $\sum a_n$ or $\sum b_n$. In some cases, we may find that changing how we label the series may help us simplify the limit.

However, to be consistent in these notes, we will label the given series as $\sum a_n$.

Example 2: Use the Limit Comparison Test to determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$$

We will compare this series to the *p*-series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$.

Note that both the original series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$ and new series $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ consist of positive terms.

In the series $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$, $p = \frac{1}{2}$ and $0 < \frac{1}{2} \leq 1$. Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges.

Let $a_n = \frac{1}{\sqrt{n+4}}$ and $b_n = \frac{1}{\sqrt{n}}$.

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty} \frac{1}{\sqrt{n+4}} \div \frac{1}{\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n+4}} \cdot \frac{\sqrt{n}}{1} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+4}} = 1$$

Since $L = 1$ and 1 is finite and positive, both series diverge.

Example 3: Use the Limit Comparison Test to determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{3n^2+5n-1}{4n^7+8n^3+9}$$

We will compare this series to the *p*-series $\sum_{n=1}^{\infty} \frac{n^2}{n^7} = \sum_{n=1}^{\infty} \frac{1}{n^5}$.

Note that both the original series $\sum_{n=1}^{\infty} \frac{3n^2+5n-1}{4n^7+8n^3+9}$ and new series $\sum_{n=1}^{\infty} \frac{1}{n^5}$ consist of positive terms.

In the series $\sum_{n=1}^{\infty} \frac{1}{n^5}$, $p=5$ and $5>1$. Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^5}$ converges.

Let $a_n = \frac{3n^2+5n-1}{4n^7+8n^3+9}$ and $b_n = \frac{1}{n^5}$.

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty} \frac{3n^2+5n-1}{4n^7+8n^3+9} \div \frac{1}{n^5} = \lim_{n\to\infty} \frac{3n^2+5n-1}{4n^7+8n^3+9} \cdot \frac{n^5}{1} = \lim_{n\to\infty} \frac{3n^7+5n^6-n^5}{4n^7+8n^3+9} = \frac{3}{4}$$

Since $L = \frac{3}{4}$ and $\frac{3}{4}$ is finite and positive, both series converge.

Example 4: Use the Limit Comparison Test to determine whether the series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{5^n}{2^n + 7}$$

We will compare this series to the geometric series $\sum_{n=0}^{\infty} \frac{5^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{5}{2}\right)^n$.

Note that both the original series $\sum_{n=0}^{\infty} \frac{5^n}{2^n + 7}$ and new series $\sum_{n=0}^{\infty} \left(\frac{5}{2}\right)^n$ consist of positive terms.

In the series $\sum_{n=0}^{\infty} \left(\frac{5}{2}\right)^n$, $r = \frac{5}{2}$ and $\left|\frac{5}{2}\right| \geq 1$. Therefore, $\sum_{n=0}^{\infty} \left(\frac{5}{2}\right)^n$ diverges.

Let $a_n = \frac{5^n}{2^n + 7}$ and $b_n = \left(\frac{5}{2}\right)^n$.

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty} \frac{\frac{5^n}{2^n + 7}}{\left(\frac{5}{2}\right)^n} = \lim_{n\to\infty} \frac{5^n}{2^n + 7} \cdot \frac{2^n}{5^n} = \lim_{n\to\infty} \frac{2^n}{2^n + 7} = 1$$

Since $L = 1$ and 1 is finite and positive, both series diverge.

Example 5: Use the Limit Comparison Test to determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{3^n}{4^n - 1}$$

We will compare this series to the geometric series $\sum_{n=1}^{\infty} \frac{3^n}{4^n} = \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$.

Note that both the original series $\sum_{n=1}^{\infty} \frac{3^n}{4^n - 1}$ and new series $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$ consist of positive terms.

In the series $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$, $r = \frac{3}{4}$ and $\left|\frac{3}{4}\right| < 1$. Therefore, $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$ converges.

Let $a_n = \frac{3^n}{4^n - 1}$ and $b_n = \left(\frac{3}{4}\right)^n$.

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty} \frac{\frac{3^n}{4^n - 1}}{\left(\frac{3}{4}\right)^n} = \lim_{n\to\infty} \frac{3^n}{4^n - 1} \cdot \frac{4^n}{3^n} = \lim_{n\to\infty} \frac{4^n}{4^n - 1} = 1$$

Since $L = 1$ and 1 is finite and positive, both series converge.

Activity Sheet with Answers

Activity: Limit Comparison Test Name: ________________________

1. The Limit Comparison Test requires us to calculate $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)=L$. What must be true about L in order to be able to apply this test?

L must be positive and finite.

Determine whether each of the following series converge or diverge using the Limit Comparison Test.

2. $\sum_{n=1}^{\infty}\frac{5}{\sqrt[4]{n^3+8}}$

We will compare this series to the p-series $\sum_{n=1}^{\infty}\frac{1}{\sqrt[4]{n^3}}=\sum_{n=1}^{\infty}\frac{1}{n^{3/4}}$.

Note that both the original series $\sum_{n=1}^{\infty}\frac{5}{\sqrt[4]{n^3+8}}$ and new series $\sum_{n=1}^{\infty}\frac{1}{n^{3/4}}$ consist of positive terms.

In the series $\sum_{n=1}^{\infty}\frac{1}{n^{3/4}}$, $p=\frac{3}{4}$ and $0<\frac{3}{4}\le 1$. Therefore, $\sum_{n=1}^{\infty}\frac{1}{n^{3/4}}$ diverges.

Let $a_n=\frac{5}{\sqrt[4]{n^3+8}}$ and $b_n=\frac{1}{n^{3/4}}$.

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)=\lim_{n\to\infty}\frac{5}{\sqrt[4]{n^3+8}}\div\frac{1}{\sqrt[4]{n^3}}=\lim_{n\to\infty}\frac{5}{\sqrt[4]{n^3+8}}\cdot\frac{\sqrt[4]{n^3}}{1}=\lim_{n\to\infty}\frac{5\sqrt[4]{n^3}}{\sqrt[4]{n^3+8}}=5$$

Since $L=5$ and 5 is finite and positive, both series diverge.

3. $\sum_{n=0}^{\infty} \frac{8^n + 4}{13^n + 7}$

We will compare this series to the geometric series $\sum_{n=0}^{\infty} \frac{8^n}{13^n} = \sum_{n=0}^{\infty} \left(\frac{8}{13}\right)^n$.

Note that both the original series $\sum_{n=0}^{\infty} \frac{8^n + 4}{13^n + 7}$ and new series $\sum_{n=0}^{\infty} \left(\frac{8}{13}\right)^n$ consist of positive terms.

In the series $\sum_{n=0}^{\infty} \left(\frac{8}{13}\right)^n$, $r = \frac{8}{13}$ and $\left|\frac{8}{13}\right| < 1$. Therefore, $\sum_{n=0}^{\infty} \left(\frac{8}{13}\right)^n$ converges.

Let $a_n = \frac{8^n + 4}{13^n + 7}$ and $b_n = \left(\frac{8}{13}\right)^n$.

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty} \frac{\frac{8^n + 4}{13^n + 7}}{\left(\frac{8}{13}\right)^n} = \lim_{n\to\infty} \frac{8^n + 4}{13^n + 7} \cdot \frac{13^n}{8^n} = \lim_{n\to\infty} \frac{13^n \left(8^n + 4\right)}{8^n \left(13^n + 7\right)} = 1$$

Since $L = 1$ and 1 is finite and positive, both series converge.

4. $\sum_{n=1}^{\infty} \frac{8n^2+3n-7}{9n^3-2n}$

We will compare this series to the harmonic series $\sum_{n=1}^{\infty} \frac{n^2}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n}$.

Note that both the original series $\sum_{n=1}^{\infty} \frac{8n^2+3n-7}{9n^3-2n}$ and new series $\sum_{n=1}^{\infty} \frac{1}{n}$ consist of positive terms.

We know that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Let $a_n = \frac{8n^2+3n-7}{9n^3-2n}$ and $b_n = \frac{1}{n}$.

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty} \frac{8n^2+3n-7}{9n^3-2n} \div \frac{1}{n} = \lim_{n\to\infty} \frac{8n^2+3n-7}{9n^3-2n} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{8n^3+3n^2-7n}{9n^3-2n} = \frac{8}{9}$$

Since $L = \frac{8}{9}$ and $\frac{8}{9}$ is finite and positive, both series diverge.

5. When asked to determine if a series converges or diverges, what would lead you to try the Limit Comparison Test rather than other methods?

Sample Answer: If the series resembled a geometric series or *p*-series, but did not exactly match the form of either series type, I would try Limit Comparison Test. I could also try to apply the Direct Comparison Test, but then I would need to determine which series had larger or smaller terms. With the Limit Comparison Test, I do not need to know which is larger or smaller. I can find $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)$ and if the limit is positive and finite, I can make a conclusion about the given series.

Let's assume $a_n > 0$ and $b_n > 0$. You have done the work to calculate the following limits. If you arrived at each of the following conclusions, would you be able to apply the Limit Comparison Test? Why or why not?

6. $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \infty$

We cannot apply the Limit Comparison Test. The limit is infinite. In the Limit Comparison Test, the limit must be finite.

7. $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{1}{3}$

We can apply the Limit Comparison Test. The limit is $\frac{1}{3}$. This is positive and finite.

8. $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = 1$

We can apply the Limit Comparison Test. The limit is 1. This is positive and finite.

9. $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = 0$

We cannot apply the Limit Comparison Test. The limit is 0. This is not a positive value.

10. $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \sqrt{7}$

We can apply the Limit Comparison Test. The limit is $\sqrt{7}$. This is positive and finite.

Lesson 7: Ratio Test

Teacher Notes

Overview:

In this lesson, students will learn about the Ratio Test. To apply this test, students will take the limit of the ratio of two terms from the same series. This test does not require students to apply information they've previously learned regarding geometric series or *p*-series. The Ratio Test is useful when working with series involving factorials, as well as series that combine different types of functions.

The Ratio Test will be extremely useful for students who continue to study calculus. For example, they will use this test to discover the radius of convergence for Taylor and Maclaurin polynomial approximations.

The activity for this lesson requires students to apply the Ratio Test to several series as well as describe the conditions of the Ratio Test. Additionally, students are asked to simplify an expression involving factorials. As the final activity, students are given several different series. For each series, they must identify an appropriate convergence/divergence test.

Objectives:

- Find the limit of a ratio
- Apply the Ratio Test

Standards:

II. Limits: LIM-7.A.11: The ratio test is a method to determine whether a series of numbers converges or diverges.

Guided Notes with Answers

Ratio Test Name: ________________________

Example 1: We are given the series $\sum_{n=1}^{\infty} \frac{n}{5^n}$ and asked to determine whether the series converges or diverges. What methods might we try in order to answer this question?

1. The n^{th}-Term Test for Divergence

   $a_n = \frac{n}{5^n}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{5^n} = 0$ by the Law of Dominance

   Since $\lim_{n\to\infty} a_n = 0$, we cannot make a conclusion about whether $\sum_{n=1}^{\infty} \frac{n}{5^n}$ diverges. The n^{th}-Term Test is inconclusive for this series.

2. Geometric Series

   The series $\sum_{n=1}^{\infty} \frac{n}{5^n}$ is not of the form $\sum_{n=0}^{\infty} ar^n$. Additionally, we cannot factor out a value to adjust the given series to fit the form. This is not a geometric series. We cannot use what we know about geometric series to determine whether the series converges or diverges.

3. *p*-Series

   The series $\sum_{n=1}^{\infty} \frac{n}{5^n}$ does not fit the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$. This is not a *p*-series. We cannot use what we know about *p*-series to determine whether the series converges or diverges.

4. Direct Comparison Test and Limit Comparison Test

   The series $\sum_{n=1}^{\infty} \frac{n}{5^n}$ is not similar to a geometric series or a *p*-series. We do not have a known series to use in a comparison. We cannot make a conclusion about this series using the Direct Comparison Test or the Limit Comparison Test.

None of the methods we have studied so far is applicable to this particular series.

However, there is another test we can use to determine the convergence or divergence of a series: The Ratio Test.

The Ratio Test is useful since we do not need to compare the given series to another series. We only need to use the given series and the expression for its terms. We can also use the Ratio Test for series with negative terms.

Ratio Test

Let $\sum a_n$ be a series with nonzero terms.

1. The series $\sum a_n$ converges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$.
2. The series $\sum a_n$ diverges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\infty$.
3. The Ratio Test is inconclusive if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ does not exist.

The absolute values are important, especially for series with some positive terms and some negative terms, but where the positive and negative values do not alternate from one term to the next. In a case such as this, the limit of the ratio of successive terms with the absolute value would exist, but the limit without the absolute value would not exist. We will not investigate a case such as this in the notes but interested students can ask to see an example later.

Note: For interested students, explore the following series:

$$\sum_{n=0}^{\infty}\frac{\sin\left(\frac{\pi}{4}+\frac{n\pi}{2}\right)}{2^n}=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{4}+\left(-\frac{\sqrt{2}}{8}\right)+\left(-\frac{\sqrt{2}}{16}\right)+\frac{\sqrt{2}}{64}+\cdots$$

$$=\sqrt{2}\left(\frac{1}{2}+\frac{1}{4}+\left(-\frac{1}{8}\right)+\left(-\frac{1}{16}\right)+\frac{1}{64}+\cdots\right).$$

The Ratio Test requires us to find $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$. Unlike the Limit Comparison Test, we cannot calculate the limit of the reciprocal and arrive at the same conclusion. For example, if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 5$, then $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \frac{1}{5}$. This would lead to two different conclusions using the Ratio Test.

Let's apply the Ratio Test to the series in **Example 1**: $\sum_{n=1}^{\infty}\frac{n}{5^n}$.

Note that $\sum_{n=1}^{\infty}\frac{n}{5^n}$ has nonzero terms.

Let $a_n = \frac{n}{5^n}$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{n+1}{5^{n+1}} \div \frac{n}{5^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{n+1}{5^{n+1}} \cdot \frac{5^n}{n}\right| \\
&= \lim_{n\to\infty}\left|\frac{5^n}{5^{n+1}} \cdot \frac{n+1}{n}\right| \\
&= \lim_{n\to\infty}\left|\frac{5^n}{5\cdot 5^n} \cdot \frac{n+1}{n}\right| \\
&= \lim_{n\to\infty}\left|\frac{1}{5} \cdot \frac{n+1}{n}\right| \\
&= \frac{1}{5}
\end{aligned}$$

Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{5}$ and $\frac{1}{5} < 1$, the series $\sum_{n=1}^{\infty}\frac{n}{5^n}$ converges by the Ratio Test.

Example 2: Use the Ratio Test to determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n$$

Note that $\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n$ has nonzero terms.

Let $a_n = n\left(\frac{2}{3}\right)^n$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\left(\frac{2}{3}\right)^{n+1}}{n\left(\frac{2}{3}\right)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)(2)^{n+1}}{(3)^{n+1}} \div \frac{n(2)^n}{(3)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)(2)^{n+1}}{(3)^{n+1}} \cdot \frac{(3)^n}{n(2)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2^{n+1}}{2^n} \cdot \frac{3^n}{3^{n+1}} \cdot \frac{n+1}{n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2\cdot 2^n}{2^n} \cdot \frac{3^n}{3\cdot 3^n} \cdot \frac{n+1}{n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2}{1} \cdot \frac{1}{3} \cdot \frac{n+1}{n}\right|$$

$$= \frac{2}{3}$$

Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{2}{3}$ and $\frac{2}{3} < 1$, the series $\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n$ converges by the Ratio Test.

Example 3: Use the Ratio Test to determine whether the series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{8^n}{n^6}$$

Note that $\sum_{n=2}^{\infty} \frac{8^n}{n^6}$ has nonzero terms.

Let $a_n = \frac{8^n}{n^6}$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{8^{n+1}}{(n+1)^6} \div \frac{8^n}{n^6}\right| \\
&= \lim_{n\to\infty}\left|\frac{8^{n+1}}{(n+1)^6} \cdot \frac{n^6}{8^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{8^{n+1}}{8^n} \cdot \frac{n^6}{(n+1)^6}\right| \\
&= \lim_{n\to\infty}\left|\frac{8\cdot 8^n}{8^n} \cdot \left(\frac{n}{n+1}\right)^6\right| \\
&= \lim_{n\to\infty}\left|\frac{8}{1} \cdot \left(\frac{n}{n+1}\right)^6\right| \\
&= 8
\end{aligned}$$

Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 8$ and $8 > 1$, the series $\sum_{n=2}^{\infty} \frac{8^n}{n^6}$ diverges by the Ratio Test.

Example 4: Use the Ratio Test to determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{6}{n}$$

Note that $\sum_{n=1}^{\infty} \frac{6}{n}$ has nonzero terms.

Let $a_n = \frac{6}{n}$.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{6}{n+1} \div \frac{6}{n}\right| \\ &= \lim_{n\to\infty}\left|\frac{6}{n+1} \cdot \frac{n}{6}\right| \\ &= \lim_{n\to\infty}\left|\frac{6n}{6(n+1)}\right| \\ &= \lim_{n\to\infty}\left|\frac{n}{n+1}\right| \\ &= 1\end{aligned}$$

Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive for the series $\sum_{n=1}^{\infty} \frac{6}{n}$.

However, we can identify $\sum_{n=1}^{\infty} \frac{6}{n}$ as a multiple of the harmonic series. Therefore, we can conclude that $\sum_{n=1}^{\infty} \frac{6}{n}$ diverges by other means.

Example 5: Use the Ratio Test to determine whether the series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{3}{n^4+1}$$

Note that $\sum_{n=2}^{\infty} \frac{3}{n^4+1}$ has nonzero terms.

Let $a_n = \frac{3}{n^4+1}$.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{3}{(n+1)^4+1} \div \frac{3}{n^4+1} \right| \\
&= \lim_{n\to\infty} \left| \frac{3}{(n+1)^4+1} \cdot \frac{n^4+1}{3} \right| \\
&= \lim_{n\to\infty} \left| \frac{n^4+1}{(n+1)^4+1} \right| \\
&= 1
\end{aligned}$$

Since $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the Ratio Test is inconclusive for the series $\sum_{n=2}^{\infty} \frac{3}{n^4+1}$.

However, we can compare $\sum_{n=2}^{\infty} \frac{3}{n^4+1}$ to the series $\sum_{n=2}^{\infty} \frac{3}{n^4}$. Both series consist of positive terms, and the terms of $\sum_{n=2}^{\infty} \frac{3}{n^4+1}$ are less than the terms of $\sum_{n=2}^{\infty} \frac{3}{n^4}$. Since we know $\sum_{n=2}^{\infty} \frac{3}{n^4}$ is a convergent *p*-series, we can conclude that $\sum_{n=2}^{\infty} \frac{3}{n^4+1}$ converges as well by the Direct Comparison Test.

Example 6: Use the Ratio Test to determine whether the series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{9^n}{n!}$$

Note that $\sum_{n=2}^{\infty} \frac{9^n}{n!}$ has nonzero terms.

Let $a_n = \frac{9^n}{n!}$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{9^{n+1}}{(n+1)!} \div \frac{9^n}{n!}\right| \\
&= \lim_{n\to\infty}\left|\frac{9^{n+1}}{(n+1)!} \cdot \frac{n!}{9^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{9^{n+1}}{9^n} \cdot \frac{n!}{(n+1)!}\right| \\
&= \lim_{n\to\infty}\left|\frac{9 \cdot 9^n}{9^n} \cdot \frac{n!}{(n+1) \cdot n!}\right| \\
&= \lim_{n\to\infty}\left|\frac{9}{1} \cdot \frac{1}{(n+1)}\right| \\
&= 0
\end{aligned}$$

Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$ and $0 < 1$, the series $\sum_{n=2}^{\infty} \frac{9^n}{n!}$ converges by the Ratio Test.

Example 7: Use the Ratio Test to determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{(n-1)!}{n^7}$$

Note that $\sum_{n=1}^{\infty} \frac{(n-1)!}{n^7}$ has nonzero terms.

Let $a_n = \frac{(n-1)!}{n^7}$.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(n+1-1)!}{(n+1)^7} \div \frac{(n-1)!}{n^7} \right| \\
&= \lim_{n\to\infty} \left| \frac{n!}{(n+1)^7} \cdot \frac{n^7}{(n-1)!} \right| \\
&= \lim_{n\to\infty} \left| \frac{n!}{(n-1)!} \cdot \frac{n^7}{(n+1)^7} \right| \\
&= \lim_{n\to\infty} \left| \frac{n \cdot (n-1)!}{(n-1)!} \cdot \left(\frac{n}{n+1} \right)^7 \right| \\
&= \lim_{n\to\infty} \left| \frac{n}{1} \cdot \left(\frac{n}{n+1} \right)^7 \right| \\
&= \infty
\end{aligned}$$

Since $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, the series $\sum_{n=1}^{\infty} \frac{(n-1)!}{n^7}$ diverges by the Ratio Test.

Example 8: Use the Ratio Test to determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} n\left(\frac{-9}{7}\right)^n$$

Note that $\sum_{n=1}^{\infty} n\left(\frac{-9}{7}\right)^n$ has nonzero terms.

Let $a_n = n\left(\frac{-9}{7}\right)^n$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)(-9)^{n+1}}{7^{n+1}} \div \frac{n(-9)^n}{7^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)(-9)^{n+1}}{7^{n+1}} \cdot \frac{7^n}{n(-9)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(-9)^{n+1}}{(-9)^n} \cdot \frac{7^n}{7^{n+1}} \cdot \frac{n+1}{n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(-9)\cdot(-9)^n}{(-9)^n} \cdot \frac{7^n}{7\cdot 7^n} \cdot \frac{n+1}{n}\right| \\
&= \lim_{n\to\infty}\left|\frac{-9}{1} \cdot \frac{1}{7} \cdot \frac{n+1}{n}\right| \\
&= \frac{9}{7}
\end{aligned}$$

Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{9}{7}$ and $\frac{9}{7} > 1$, the series $\sum_{n=1}^{\infty} n\left(\frac{-9}{7}\right)^n$ diverges by the Ratio Test.

Activity Sheet with Answers

Activity: Ratio Test Name: ________________________

Complete the following statements regarding the Ratio Test.

1. The series $\sum a_n$ must have nonzero terms.

2. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$, then $\sum a_n$ converges.

3. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\infty$, then $\sum a_n$ diverges.

4. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$, the Ratio Test is inconclusive.

5. Describe an advantage of using the Ratio Test instead of the Direct or Limit Comparison Tests.

With the Ratio Test, we do not need to use an additional series to make a conclusion about the given series. Both the Direct Comparison Test and Limit Comparison Test require us to identify a series that is similar to the given series.

6. Simplify the following factorial expression: $\frac{n!}{(n-5)!}$.

$$\frac{n!}{(n-5)!}=\frac{n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdot(n-4)\cdot(n-5)!}{(n-5)!}$$
$$=n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdot(n-4)$$

7. Simplify the following factorial expression: $\frac{(4n-3)!}{(4n+2)!}$.

$$\frac{(4n-3)!}{(4n+2)!}=\frac{(4n-3)!}{(4n+2)\cdot(4n+1)\cdot(4n)\cdot(4n-1)\cdot(4n-2)\cdot(4n-3)!}$$
$$=\frac{1}{(4n+2)\cdot(4n+1)\cdot(4n)\cdot(4n-1)\cdot(4n-2)}$$

Determine whether each of the following series converges or diverges using the Ratio Test.

8. $\sum_{n=1}^{\infty} n(n!)$

Note that $\sum_{n=1}^{\infty} n(n!)$ has nonzero terms.

Let $a_n = n(n!)$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(n+1)(n+1)!}{(n)(n!)}\right|$$
$$=\lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\frac{(n+1)!}{n!}\right|$$
$$=\lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\frac{(n+1)\cdot n!}{n!}\right|$$
$$=\lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\frac{n+1}{1}\right|$$
$$=\infty$$

Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\infty$, the series $\sum_{n=1}^{\infty} n(n!)$ diverges by the Ratio Test.

9. $\sum_{n=1}^{\infty} \frac{n^2}{(3n)!}$

Note that $\sum_{n=1}^{\infty} \frac{n^2}{(3n)!}$ has nonzero terms.

Let $a_n = \frac{n^2}{(3n)!}$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)^2}{(3n+3)!} \div \frac{n^2}{(3n)!}\right| \\
&= \lim_{n\to\infty}\left|\frac{(n+1)^2}{(3n+3)!} \cdot \frac{(3n)!}{n^2}\right| \\
&= \lim_{n\to\infty}\left|\frac{(3n)!}{(3n+3)!} \cdot \frac{(n+1)^2}{n^2}\right| \\
&= \lim_{n\to\infty}\left|\frac{(3n)!}{(3n+3)(3n+2)(3n+1)(3n)!} \cdot \left(\frac{n+1}{n}\right)^2\right| \\
&= \lim_{n\to\infty}\left|\frac{1}{(3n+3)(3n+2)(3n+1)} \cdot \left(\frac{n+1}{n}\right)^2\right| \\
&= 0
\end{aligned}$$

Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$ and $0 < 1$, the series $\sum_{n=1}^{\infty} \frac{n^2}{(3n)!}$ converges by the Ratio Test.

10. $\sum_{n=3}^{\infty} \frac{2n^2+3n}{4^n}$

Note that $\sum_{n=3}^{\infty} \frac{2n^2+3n}{4^n}$ has nonzero terms.

Let $a_n = \frac{2n^2+3n}{4^n}$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{2(n+1)^2+3n+3}{4^{n+1}} \div \frac{2n^2+3n}{4^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{2(n+1)^2+3n+3}{4^{n+1}} \cdot \frac{4^n}{2n^2+3n}\right| \\
&= \lim_{n\to\infty}\left|\frac{2(n+1)^2+3n+3}{2n^2+3n} \cdot \frac{4^n}{4^{n+1}}\right| \\
&= \lim_{n\to\infty}\left|\frac{2(n+1)^2+3n+3}{2n^2+3n} \cdot \frac{4^n}{4\cdot 4^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{2(n+1)^2+3n+3}{2n^2+3n} \cdot \frac{1}{4}\right| \\
&= \frac{1}{4}
\end{aligned}$$

Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{4}$ and $\frac{1}{4} < 1$, the series $\sum_{n=3}^{\infty} \frac{2n^2+3n}{4^n}$ converges by the Ratio Test.

11. $\sum_{n=3}^{\infty}(n-2)\left(\frac{7}{5}\right)^n$

Note that $\sum_{n=3}^{\infty}(n-2)\left(\frac{7}{5}\right)^n$ has nonzero terms.

Let $a_n=(n-2)\left(\frac{7}{5}\right)^n$.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{((n+1)-2)(7^{n+1})}{5^{n+1}}\cdot\frac{5^n}{(n-2)(7^n)}\right| \\ &= \lim_{n\to\infty}\left|\frac{(n-1)}{(n-2)}\cdot\frac{7^{n+1}}{7^n}\cdot\frac{5^n}{5^{n+1}}\right| \\ &= \lim_{n\to\infty}\left|\frac{(n-1)}{(n-2)}\cdot\frac{7\cdot 7^n}{7^n}\cdot\frac{5^n}{5\cdot 5^n}\right| \\ &= \lim_{n\to\infty}\left|\frac{(n-1)}{(n-2)}\cdot\frac{7}{1}\cdot\frac{1}{5}\right| \\ &= \frac{7}{5}\end{aligned}$$

Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\frac{7}{5}$ and $\frac{7}{5}>1$, the series $\sum_{n=3}^{\infty}(n-2)\left(\frac{7}{5}\right)^n$ diverges by the Ratio Test.

We could have also determined that $\sum_{n=3}^{\infty}(n-2)\left(\frac{7}{5}\right)^n$ diverges using the Direct Comparison Test and comparing it to the divergent geometric series $\sum_{n=3}^{\infty}\left(\frac{7}{5}\right)^n$. The Direct Comparison Test may be preferable since we can quickly identify that for $n\geq 3$ $\left(\frac{7}{5}\right)^n\leq(n-2)\left(\frac{7}{5}\right)^n$ and therefore conclude that $\sum_{n=3}^{\infty}(n-2)\left(\frac{7}{5}\right)^n$ diverges without having to find $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ using the Ratio Test.

For each of the following, match the series with the method you would use to determine convergence. Some series can be determined using more than one method. However, you may list each method only once.

12. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[8]{n^7}}$ C

13. $\sum_{n=1}^{\infty} \frac{(n+2)!}{11}$ F

14. $\sum_{n=1}^{\infty} \frac{8^n}{13^n - 9}$ E

15. $\sum_{n=1}^{\infty} \frac{4}{3^n}$ B

16. $\sum_{n=1}^{\infty} \frac{9n-2}{7n+1}$ A

17. $\sum_{n=1}^{\infty} \frac{6}{n^2+1}$ D

A. The n^{th}-Term Test for Divergence

B. Geometric series

C. p-series

D. Direct Comparison Test

E. Limit Comparison Test

F. Ratio Test

Appendix: Guided Notes and Activity Sheets for Students

Guided Notes

Introduction to Sequences and Series Name: ________________________

Definitions

Sequence:

We use subscript notation to represent the terms of the sequence: ________________

The n^{th} term is denoted by ____.

The entire sequence is denoted by ____.

Series: _____________________________

We can calculate partial sums as well as some infinite sums.

____________ are denoted by S_m, where the first m terms of the sequence are added.

Notation using sigma: $S_m = \sum_{n=1}^{m} a_n = a_1 + a_2 + a_3 + \cdots + a_m$

If the limit L of a sequence exists as n goes to infinity, then the sequence _________ to L.

If the limit of a sequence does not exist as n goes to infinity, then the sequence ________.

Example 1: Write the first 5 terms of the sequence whose n^{th} term is $a_n = \frac{3^n}{n!}$.

Example 2: Find an expression for the n^{th} term of the sequence $3,\ 7,\ 11,\ 15,\ \ldots$.

Example 3: Find an expression for the n^{th} term of the sequence $1,\ -\frac{1}{2},\ \frac{1}{6},\ -\frac{1}{24},\ \ldots$.

Example 4: Determine if the sequence converges or diverges by finding the limit, if possible: $a_n = 5 - \frac{1}{n^3}$.

Example 5: Determine if the sequence converges or diverges by finding the limit, if possible: $a_n = \cos\left(\frac{2}{n}\right)$.

Example 6: Determine if the sequence converges or diverges by finding the limit, if possible: $a_n = 1 + (-1)^n$.

Example 7: Determine if the sequence converges or diverges by finding the limit, if possible: $a_n = \sin(\pi n)$.

Example 8: Find the partial sum S_4 given $a_n = \dfrac{3n}{n+2}$.

Activity Sheet

Activity: Introduction to Sequences and Series Name: ________________________

1. Classify each of the following as a sequence or a series.

a) $\frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ ...$

b) $\frac{4}{5}+\frac{8}{125}+\frac{16}{625}+\frac{32}{3125}+\cdots$

c) $\frac{1}{7}+\frac{2}{7}+\frac{6}{7}+\frac{24}{7}+\cdots$

d) $-\frac{3}{8},\ \frac{3}{16},\ -\frac{3}{32},\ \frac{3}{64},\ ...$

2. Create your own pattern for a sequence. Write the expression for the n^{th} term, using correct notation. Then list the first five terms of your sequence.

3. For each of the following, write an expression for the n^{th} term of the sequence.

a) $-\frac{5}{3},\ \frac{25}{3},\ -\frac{125}{3},\ \frac{625}{3},\ -\frac{3125}{3},\ ...$

b) $27,\ \frac{27}{2},\ \frac{9}{2},\ \frac{9}{8},\ \frac{9}{40},\ ...$

c) $1, 6, 11, 16, 21, ...$

d) $-\frac{2}{7}, -\frac{6}{49}, -\frac{24}{343}, -\frac{120}{2401}, -\frac{720}{16807}, ...$

Find the first five terms of each of the following sequences. Then determine if the sequence converges or diverges. If the sequence converges, find its limit.

4. $a_n = \frac{5n}{3+n}$

5. $a_n = (-1)^n n!$

6. $a_n = \dfrac{5^n}{n^3}$

7. Find the partial sum S_6 of the series $-4+(-2)+\left(-\dfrac{2}{3}\right)+\left(-\dfrac{1}{6}\right)+\cdots$

Guided Notes

The n^{th}-Term Test for Divergence Name: ________________________

Recall from previous lesson

Series: ____________________________

We can calculate partial sums as well as some infinite sums.

Notation using sigma: $S_m = \sum_{n=1}^{m} a_n = a_1 + a_2 + a_3 + \cdots + a_m$

Example 1: Consider the series $0+0+0+0+\cdots$.

If we let a_n represent the n^{th} term of this series, what is $\lim_{n\to\infty} a_n$?

If we added infinitely many terms of this series, what do you think the sum would be?

Based on our answers above, do you think this series converges or diverges?

Example 2: Consider the series $1+\frac{1}{2}+\frac{1}{2}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\cdots$.

If we let a_n represent the n^{th} term of this series, what is $\lim_{n\to\infty} a_n$?

If we added infinitely many terms of this series, what do you think the sum would be?

Based on our answers above, do you think this series converges or diverges?

Example 3: Consider the series $5+7+9+11+13+\cdots$.

If we let a_n represent the n^{th} term of this series, what is $\lim_{n\to\infty} a_n$?

If we added infinitely many terms of this series, what do you think the sum would be?

Based on our answers above, do you think this series converges or diverges?

Example 4: Consider the series $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\cdots$.

If we let a_n represent the n^{th} term of this series, what is $\lim_{n\to\infty} a_n$?

If we added infinitely many terms of this series, what do you think the sum would be?

Based on our answers above, do you think this series converges or diverges?

Based on our examples, what conclusion can we make regarding $\lim_{n\to\infty} a_n$ and the convergence or divergence of a series?

This leads us to our first test to determine whether a series diverges.

Think about the following:

If a series converges, the limit of its n^{th} term must be 0.

That is, if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

It's generally easier to determine the limit of the n^{th} term than it is to determine whether a series converges. Therefore, we will use the ______________ of the statement above.

The n^{th}-Term Test for Divergence

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.

Note: If $\lim_{n\to\infty} a_n = 0$, we cannot make any conclusion about the series. Look back at **Example 1** and **Example 2** to verify this.

Additionally, notice that when we express a sum as _____ without ________, we always mean _____________, starting at a finite value of *n.* Since the series has _____________ _____, the convergence or divergence of the series does not depend on the ___________ ____________________. However, the ________________________will depend on the initial value of *n.*

For example, let's look at $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \sum_{n=3}^{\infty} a_n$. Since a_1 and a_2 are __________,

$\sum_{n=1}^{\infty} a_n$ ________________ if and only if $\sum_{n=3}^{\infty} a_n$ ________________.

Example 5: Use the n^{th}-Term Test for Divergence to determine whether the series diverges.

$$\sum_{n=0}^{\infty} 5(1.23)^n$$

Example 6: Use the n^{th}-Term Test for Divergence to determine whether the series diverges.

$$\sum_{n=3}^{\infty} \frac{7}{n(n+8)}$$

Example 7: Use the n^{th}-Term Test for Divergence to determine whether the series diverges.

$$\sum_{n=1}^{\infty} \frac{n}{2n+3}$$

Example 8: Use the n^{th}-Term Test for Divergence to determine whether the series diverges.

$$\sum_{n=1}^{\infty} \frac{3^n}{n!}$$

Example 9: Use the n^{th}-Term Test for Divergence to determine whether the series diverges.

$$\sum_{n=0}^{\infty}\left(9-\frac{1}{n^3}\right)$$

Activity Sheet

Activity: The n^{th}-Term Test for Divergence Name:______________________

Determine if each of the following statements are True or False.

1. If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ must converge.

2. If $\sum a_n$ diverges, then $\lim_{n\to\infty} a_n = 0$.

3. If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

4. If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ must diverge.

5. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ must converge.

6. If $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ must diverge.

7. If $\sum a_n$ diverges, then $\lim_{n\to\infty} a_n \neq 0$.

8. If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n \neq 0$.

Based on your results above, only statements _____ and _____ are true.

Therefore, if we can show that___________, then we can conclude that a series _______ using the n^{th}-Term Test for Divergence.

If we can show that_________, then the n^{th}-Term Test for Divergence is _____________.

9. Use a table to find $\lim_{n\to\infty} a_n$ given $a_n = \dfrac{8}{n+9}$. Round your answers to the nearest hundred thousandth.

n	1	10	100	1,000	10,000	100,000
a_n						

Based on your table, $\lim_{n\to\infty} a_n =$ _____. What conclusion can we make about $\sum \dfrac{8}{n+9}$?

10. Use a table to find $\lim_{n\to\infty} a_n$ given $a_n = \dfrac{(n+1)!}{n!}$.

n	1	10	100	1,000	10,000	100,000
a_n						

Based on your table, $\lim_{n\to\infty} a_n =$ _____. What conclusion can we make about $\sum \dfrac{(n+1)!}{n!}$?

For each of the following, use algebraic techniques to find $\lim_{n\to\infty} a_n$. Then apply the n^{th}-Term Test if possible.

11. $\displaystyle\sum_{n=1}^{\infty} \frac{7n}{3n-1}$

12. $\sum_{n=4}^{\infty} \frac{-3n+2}{4n^2-1}$

13. $\sum_{n=0}^{\infty} 3\left(\frac{1}{4}\right)^n$

14. $\sum_{n=2}^{\infty} \frac{-5n^3+2n}{8n^2+9n-1}$

15. $\sum_{n=1}^{\infty} 7$

16. $\sum_{n=3}^{\infty} \left(-\frac{6}{n}+5\right)$

17. $\sum_{n=4}^{\infty} (-1)^n \left(\frac{3n}{5n-6}\right)$

Guided Notes

Geometric Series Name: ________________________

Consider the series $3+\frac{3}{2}+\frac{3}{4}+\frac{3}{8}+\frac{3}{16}+\cdots$.

What do you notice about the terms of this series?

Series consisting of terms with a constant ratio are called ______________. The terms of this type of series form a ____________________.

In general, the series given by $\sum_{n=0}^{\infty} ar^n = a+ar+ar^2+ar^3+\cdots+ar^n+\cdots$ with $a \neq 0$ and $r \neq 0$ is a ______________ with ______________ r.

Example 1: Identify the ratio of the geometric series and write the series in sigma notation.

$5+10+20+40+80+\cdots$

Example 2: Identify the ratio of the geometric series and write the series in sigma notation.

$\frac{8}{5}+\frac{2}{5}+\frac{1}{10}+\frac{1}{40}+\frac{1}{160}+\cdots$

Example 3: Identify the ratio of the geometric series and write the series in sigma notation.

$1+(-3)+9+(-27)+81+\cdots$

Identifying the ratio of a geometric series is a crucial skill because the ratio determines whether the series will converge or diverge. Let's investigate this below.

You may remember from Algebra 2 that there is a formula for the sum of the terms of a finite geometric sequence: $\sum_{n=0}^{m} ar^n =$ ______________ provided that _________.

If $r=1$, then $\sum_{n=0}^{m} ar^n =$ ____.

We know that the value of an infinite series is the limit of its partial sums.

Using the formula above for $r \neq 1$, we can represent the m^{th} partial sum as

______________. Now, let's take the limit of this partial sum.

If we want this sequence of partial sums to converge, we want to find what values of r

will make ___________ exist.

If _______ or _______, $\lim_{m\to\infty} r^m$ does not exist.

When ________, ____________ and the sequence of partial sums ____________.

So assuming $|r|<1$, we can find $\lim_{m\to\infty} S_m$.

The other possibility from above was $r=1$ and $S_m = \sum_{n=0}^{m} ar^n =$ ________.

In this case, $\lim_{m\to\infty} S_m =$ ______ and therefore this geometric series would diverge.

Thus a geometric series with ratio r ______________ if $|r|<1$ and will have the sum

$\sum_{n=0}^{\infty} ar^n =$ ________.

- Notice that this characterization for the limit of the sum requires the value of the index to start at ________. We will address how to calculate sums when the index starts at other values in the examples to follow.

A geometric series with ratio r ________ if $|r| \geq 1$.

Example 4: Determine whether the geometric series converges or diverges. If the series converges, find the sum.

$$\sum_{n=0}^{\infty} 3\left(\frac{2}{7}\right)^n$$

Example 5: Determine whether the geometric series converges or diverges. If the series converges, find the sum.

$$\sum_{n=1}^{\infty} \frac{\sqrt{6}}{8^n}$$

Example 6: Determine whether the geometric series converges or diverges. If the series converges, find the sum.

$$\sum_{n=2}^{\infty}\left(-\frac{6}{11}\right)^{n}$$

Example 7: Determine whether the geometric series converges or diverges. If the series converges, find the sum.

$$\sum_{n=0}^{\infty} -\frac{1}{5}(8)^n$$

Example 8: Determine whether the geometric series converges or diverges. If the series converges, find the sum.

$$\sum_{n=1}^{\infty} \frac{e^n}{4}$$

Activity Sheet

Activity: Geometric Series Name: ________________________

1. Determine if each of the following is a geometric series. If it is a geometric series, write the series in summation notation.

a) $120+40+\frac{40}{3}+\frac{40}{9}+\frac{40}{27}+\cdots$

b) $-8+(-3)+2+7+12+\cdots$

c) $-2+4+(-8)+16+(-32)+\cdots$

d) $\pi+\pi^2+\pi^3+\pi^4+\pi^5+\cdots$

e) $-1+7+63+215+511+\cdots$

f) $10+\frac{25}{2}+\frac{125}{8}+\frac{625}{32}+\frac{3125}{128}+\cdots$

g) $1+4+9+16+25+\cdots$

h) $1000+(-250)+\frac{125}{2}+\left(-\frac{125}{8}\right)+\frac{125}{32}+\cdots$

Determine if each of the following is a geometric series. If it is a geometric series, identify the ratio r.

2. $\sum_{n=0}^{\infty} 5\left(-\frac{2}{3}\right)^n$

3. $\sum_{n=0}^{\infty} \frac{n^3}{5}$

4. $\sum_{n=1}^{\infty} -3(4)^n$

5. $\sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{9^n}{4}\right)$

6. $\sum_{n=3}^{\infty} \frac{7}{13^n}$

7. $\sum_{n=2}^{\infty} \frac{3n^4}{6^n}$

8. $\sum_{n=2}^{\infty} \frac{n!}{4+3^n}$

9. $\sum_{n=0}^{\infty} \frac{5n-1}{8n^2-4n+2}$

10. $\sum_{n=1}^{\infty} \frac{1}{3}(\pi)^n$

11. $\sum_{n=0}^{\infty} \frac{\sqrt{n^3}}{7n-9}$

12. A geometric series diverges if ________.

13. A geometric series converges if ________.

14. For a convergent geometric series, $\sum_{n=0}^{\infty} ar^n =$ ________.

15. Describe two ways to calculate the sum of a convergent geometric series $\sum_{n=3}^{\infty} ar^n$.

Determine whether each of the following geometric series converges or diverges. If the series converges, find the sum.

16. $\sum_{n=0}^{\infty} 4\left(\frac{1}{7}\right)^n$

17. $\sum_{n=0}^{\infty} 5\left(\frac{7}{3}\right)^n$

18. $\sum_{n=1}^{\infty} \frac{5}{6^n}$

19. $\sum_{n=0}^{\infty} -\frac{3}{4}\left(\sqrt{11}\right)^n$

20. $\sum_{n=2}^{\infty} -1\left(\frac{5}{6}\right)^n$

21. $\sum_{n=0}^{\infty}\left(-\frac{2}{9}\right)^{n}$

22. $\sum_{n=1}^{\infty}\frac{1}{8}\left(\sqrt{7}\right)^{n}$

23. Write a geometric series in summation notation that converges to a sum of 6.

Guided Notes

p-Series Name: ________________________

Example 1: Consider the series $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\cdots$.

What pattern do you notice in the terms of this series?

What term does not seem to fit this pattern at first glance?

Can you verify that this term matches the pattern we notice?

Example 2: Consider the series $3+\frac{3}{8}+\frac{1}{9}+\frac{3}{64}+\frac{3}{125}+\cdots$.

What pattern do you notice in the terms of this series?

What terms do not seem to fit this pattern at first glance?

Can we verify that these terms match the pattern we notice?

A series of the form $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$ is a __________ where *p* is _________ ___________.

Additionally, when _________, the series $\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is called the _________ ___________.

The series in **Example 1** and **Example 2** are both *p*-series.

We can rewrite **Example 1** as $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots =$ ___________ where __________.

It is important to notice that **Example 2** has a numerator other than 1. However, we know that infinite series are limits of finite sums. Based on properties of limits that we have previously studied, we know we can factor constants out of limits. Therefore, we can also factor constants out of infinite series.

$$\sum_{n=1}^{\infty} k a_n = k \sum_{n=1}^{\infty} a_n \text{ where } k \text{ is a constant.}$$

Therefore, **Example 2** can be rewritten as $3 + \frac{3}{8} + \frac{1}{9} + \frac{3}{64} + \frac{3}{125} + \cdots =$ ________________.

Example 3: Identify the value of *p* for the *p*-series and write the series in sigma notation.

$1 + \frac{1}{32} + \frac{1}{243} + \frac{1}{1024} + \frac{1}{3125} + \cdots$

Example 4: Identify the value of p for the p-series and write the series in sigma notation.

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{5}}+\cdots$$

Example 5: Identify the value of p for the p-series and write the series in sigma notation.

$$7+\frac{7}{\sqrt[4]{2}}+\frac{7}{\sqrt[4]{3}}+\frac{7}{\sqrt[4]{4}}+\frac{7}{\sqrt[4]{5}}+\cdots$$

Example 6: Identify the value of p for the p-series and write the series in sigma notation.

$$11+\frac{11}{2\sqrt[3]{2}}+\frac{11}{3\sqrt[3]{3}}+\frac{11}{4\sqrt[3]{4}}+\frac{11}{5\sqrt[3]{5}}+\cdots$$

Like finding the common ratio in a geometric series, identifying the value of *p* in a *p*-series is a crucial skill because *p* determines whether the series will converge or diverge.

A *p*-series with $p > 1$ will ___________.

A *p*-series with $0 < p \leq 1$ will ___________.

Remember that *p* must be ______________________.

Let's investigate why the convergent and divergent statements above are true.

Remember that an infinite series is a sequence of partial sums. When we determine whether $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges, we look at the sequence of partial sums.

Let's begin by looking at the partial sums of $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

Think back to earlier in this course. When have we added up values of a function?

The partial sums we have listed above look different from Riemann sums since each term is not being multiplied by Δx . However, if we let $\Delta x = 1$, we can rewrite the partial sums as follows:

Now, we can see that each of these partial sums can be interpreted as a _______________.

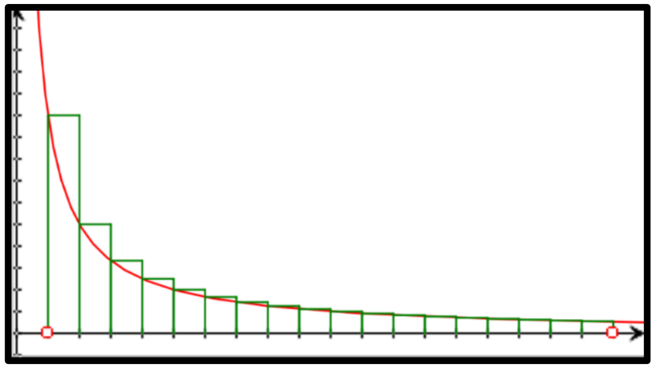
Let's look at Riemann sum approximations for $\int_1^m \frac{1}{x^p}\, dx$.

A left Riemann sum approximation with $\Delta x = 1$ would be:

$$\int_1^m \frac{1}{x^p}\, dx \approx \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ .$$

This would correspond to the partial sum ___________.

We know that _______ and thus $f(x) = \dfrac{1}{x^p}$ is a _________ function on the interval ______.



Left Riemann sum approximation [4].

Therefore, the left Riemann sum approximation is an ___________ of the integral.

That is, $S_{m-1} \geq \int_1^m \frac{1}{x^p}\,dx$.

For $0 < p < 1$,

$$\begin{aligned}
\lim_{m\to\infty} S_{m-1} &\geq \lim_{m\to\infty} \int_1^m \frac{1}{x^p}\,dx \\
&= \lim_{m\to\infty} \left. \frac{x^{(-p+1)}}{-p+1} \right|_1^m \\
&= \lim_{m\to\infty} \left(\frac{m^{(-p+1)}}{-p+1} - \frac{1^{(-p+1)}}{-p+1} \right).
\end{aligned}$$

When $0 < p < 1$, $-p+1$ is positive and therefore $m^{(-p+1)}$ grows without bound as $m \to \infty$. Thus $\lim_{m\to\infty} \left(\frac{m^{(-p+1)}}{-p+1} - \frac{1^{(-p+1)}}{-p+1} \right)$ is infinite. This implies that $\lim_{m\to\infty} S_{m-1}$ is __________ and the sequence of partial sums ___________. Thus, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ ___________.

For $p = 1$,

$$\begin{aligned}\lim_{m\to\infty} S_{m-1} &\geq \lim_{m\to\infty} \int_1^m \frac{1}{x^1}\,dx \\ &= \lim_{m\to\infty} \ln|x|\Big|_1^m \\ &= \lim_{m\to\infty} \left(\ln|m| - \ln|1|\right) \\ &= \lim_{m\to\infty} \left(\ln|m| - 0\right) \\ &= \infty.\end{aligned}$$

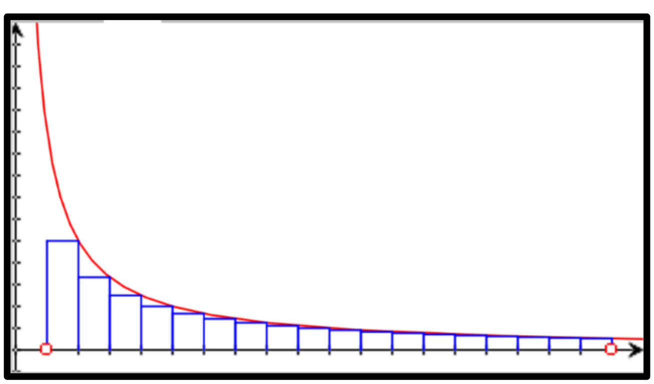
Thus $\lim_{m\to\infty} S_{m-1}$ is ________ and therefore the sequence of partial sums ________. Thus, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ ___________.

A right Riemann sum approximation with $\Delta x = 1$ would be:

$$\int_1^m \frac{1}{x^p}\,dx \approx \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}.$$

This would correspond to the partial sum ___________.

We know that _______ and thus $f(x) = \frac{1}{x^p}$ is a _________ function on the interval ______.



Right Riemann sum approximation [4].

Therefore, the right Riemann sum approximation is an _______________ of the integral.

That is, $S_m - \frac{1}{1^p}(1) \le \int_1^m \frac{1}{x^p}\,dx$, so $S_m \le \frac{1}{1^p}(1) + \int_1^m \frac{1}{x^p}\,dx$.

For $p > 1$,

$$\begin{aligned}\lim_{m\to\infty} S_m &\le \lim_{m\to\infty}\left(\frac{1}{1^p}(1) + \int_1^m \frac{1}{x^p}\,dx\right)\\ &= \frac{1}{1^p}(1) + \lim_{m\to\infty}\int_1^m \frac{1}{x^p}\,dx\\ &= \frac{1}{1^p}(1) + \lim_{m\to\infty}\left.\frac{x^{(-p+1)}}{-p+1}\right|_1^m\\ &= \frac{1}{1^p}(1) + \lim_{m\to\infty}\left(\frac{m^{(-p+1)}}{-p+1} - \frac{1^{(-p+1)}}{-p+1}\right).\end{aligned}$$

Thus, $\lim_{m\to\infty} S_m \le \frac{1}{1^p}(1) + \lim_{m\to\infty}\left(\frac{m^{(-p+1)}}{-p+1} - \frac{1^{(-p+1)}}{-p+1}\right)$.

When $p > 1$, $-p+1$ is negative and therefore $m^{(-p+1)}$ converges to 0 as $m \to \infty$. Thus $\frac{1}{1^p}(1) + \lim_{m\to\infty}\left(\frac{m^{(-p+1)}}{-p+1} - \frac{1^{(-p+1)}}{-p+1}\right)$ _______. This implies that $\lim_{m\to\infty} S_m$ _______. Since the limit of partial sums ___________, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ ___________.

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when ___________ and diverges when ___________.

We will usually assign *p*-series an initial index value of $n = 1$, however the initial value of the index can be any finite positive value. This will not affect the convergence of a *p*-series.

Using this information, we can determine whether the series in **Examples 3** through **6** converge.

Example 3:

Example 4:

Example 5:

Example 6:

Example 7: Determine whether the *p*-series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n^{6/7}} = 1 + \frac{1}{2^{6/7}} + \frac{1}{3^{6/7}} + \frac{1}{4^{6/7}} + \frac{1}{5^{6/7}} + \cdots$$

Example 8: Determine whether the *p*-series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{9}{n^{\pi}} = 9 + \frac{9}{2^{\pi}} + \frac{9}{3^{\pi}} + \frac{9}{4^{\pi}} + \frac{9}{5^{\pi}} + \cdots$$

Example 9: Determine whether the *p*-series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{3}{\sqrt[6]{n}} = 3 + \frac{3}{\sqrt[6]{2}} + \frac{3}{\sqrt[6]{3}} + \frac{3}{\sqrt[6]{4}} + \frac{3}{\sqrt[6]{5}} + \cdots$$

Example 10: Determine whether the *p*-series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{8.3}{n} = 8.3 + \frac{8.3}{2} + \frac{8.3}{3} + \frac{8.3}{4} + \frac{8.3}{5} + \cdots$$

Activity Sheet

Activity: p-Series Name: ________________________

1. Determine if each of the following is a *p*-series. If it is, write the *p*-series in summation notation.

a) $1+\frac{1}{2^{-1}}+\frac{1}{3^{-1}}+\frac{1}{4^{-1}}+\frac{1}{5^{-1}}+\ldots$

b) $7+\frac{7}{2}+\frac{7}{3}+\frac{7}{4}+\frac{7}{5}+\cdots$

c) $17+\frac{17}{2\sqrt[9]{2}}+\frac{17}{3\sqrt[9]{3}}+\frac{17}{4\sqrt[9]{4}}+\frac{17}{5\sqrt[9]{5}}+\ldots$

d) $7+7\left(\frac{5}{2}\right)^{1}+7\left(\frac{5}{2}\right)^{2}+7\left(\frac{5}{2}\right)^{3}+7\left(\frac{5}{2}\right)^{4}+\cdots$

e) $1^{-5}+2^{-5}+3^{-5}+4^{-5}+5^{-5}+\cdots$

f) $4+\frac{1}{2}+\frac{4}{27}+\frac{1}{16}+\frac{4}{125}+\cdots$

Identify p in each of the following p-series. Then, determine if the series converges or diverges.

2. $\sum_{n=1}^{\infty} \frac{1}{n^{3/7}}$

3. $\sum_{n=1}^{\infty} \frac{1}{n^{e}}$

4. Write a p-series in summation notation with $p = \frac{1}{45}$ and whose first term is 9.

5. Write a p-series in summation notation with $p = 6$ and whose first term is 1.

Determine whether each of the following series converges or diverges. Use any method we have studied.

6. $\sum_{n=1}^{\infty} \frac{8+3n}{9n}$

7. $\sum_{n=2}^{\infty} 5(3)^{n}$

8. $\sum_{n=1}^{\infty} \frac{13}{n^2 \sqrt[5]{n}}$

9. $\sum_{n=0}^{\infty} \frac{-2^n}{3}$

10. $\sum_{n=2}^{\infty} \frac{5.2^n}{n^3}$

11. $\sum_{n=1}^{\infty} \frac{-3}{n^8}$

12. $\sum_{n=0}^{\infty} \pi \left(\frac{1}{6}\right)^n$

13. $\sum_{n=1}^{\infty} \frac{4}{\sqrt{n}}$

14. $\sum_{n=3}^{\infty} 16.2$

15. $\sum_{n=0}^{\infty} \frac{4^n}{3 \cdot 5^n}$

16. Complete the following chart.

	General form of series	Series diverges if …	Series converges if…
The n^{th}-Term Test for Divergence			
Geometric Series			
p-Series			

17. The n^{th}-Term Test for Divergence can only determine that a series ___________.

18. We can calculate the sum of a convergent ______________ with an initial index value of __________ using the formula ___________.

19. In a ___________ in the form $\sum_{n=m}^{\infty} \frac{1}{n^p}$, _________ must be a ____________________.

Guided Notes

Direct Comparison Test Name: ________________________

Example 1: We are given the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ and asked to determine whether the series converges or diverges. What methods might we try in order to answer this question?

1.

2.

3.

None of the methods we have studied so far is applicable to this particular series.

What if instead of matching a known series type exactly, we looked for a type that was *similar* to the series we are working with?

The series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is *similar* to ____________.

What ____________ would we say is most *similar* to $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$?

What do we know about the series we wrote above?

Let's list a few terms of the original series we were given and the new series we have compared it to.

Let's compare the two series term by term.

We can see that each term of the original series is ____________ the corresponding term of the series we selected.

Additionally, we can compare the expression for the n^{th} term for each series.

What do we notice about these expressions?

Since we know that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ ___________, what do you think this means about our original series?

This leads us to the next convergence test we will study: the Direct Comparison Test.

Direct Comparison Test

Given two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, let $0 < a_n \leq b_n$ for all n.

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

To use this test, we must find a series similar to the series we are given. We should choose a series that is either ___________ or ___________ since we can easily determine whether these series converge or diverge.

Additionally, notice that this test requires both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ to consist of positive terms. This is an important condition to verify before applying the Direct Comparison Test.

Let's investigate why this condition is important. We will assume we only need $a_n \le b_n$.

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{-1}{n}$ and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$.

It is true that $a_n \le b_n$ for all terms since every a_n is negative and every b_n is positive. We know that $\sum_{n=1}^{\infty} \frac{-1}{n}$ __________ since it is the negative of the ___________ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a _________ that _________. So even with $a_n \le b_n$, when the terms of the series are not all positive, we cannot use the convergence of $\sum_{n=1}^{\infty} b_n$ to determine the convergence of $\sum_{n=1}^{\infty} a_n$. Also even with $a_n \le b_n$, we cannot use divergence of $\sum_{n=1}^{\infty} a_n$ to determine the divergence of $\sum_{n=1}^{\infty} b_n$.

Therefore, we must be sure that we are working with series consisting of positive terms when we apply this test.

Example 2: Use the Direct Comparison Test to determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{5}{\sqrt[3]{n}}$$

Example 3: Use the Direct Comparison Test to determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{2^n}{3^n + 5}$$

Example 4: Use the Direct Comparison Test to determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{8}{n+3}$$

Example 5: Use the Direct Comparison Test to determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^5+1}}$$

Example 6: Use the Direct Comparison Test to determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{7\sqrt[4]{n} - 2}$$

Activity Sheet

Activity: Direct Comparison Test Name: ______________________

Determine if each of the following is similar to a geometric series or a *p*-series. Then provide the series you would use with the Direct Comparison Test.

1. $\sum_{n=1}^{\infty} \frac{5^n + 2}{4^n}$

2. $\sum_{n=1}^{\infty} \frac{1}{3n^3 + 1}$

3. $\sum_{n=1}^{\infty} \frac{1}{10\sqrt[7]{n^5} - 9}$

4. $\sum_{n=1}^{\infty} \frac{6^n}{11^n + 7}$

Use the words "smaller" and "larger" to complete the following statements describing the results of the Direct Comparison Test.

5. If the ___________ series converges, the ___________ series must also converge.

6. If the ___________ series diverges, the ___________ series must also diverge.

Determine whether each of the following series converges or diverges using the Direct Comparison Test.

7. $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n^6 + 12}}$

8. $\sum_{n=1}^{\infty} \frac{8^n}{7^n - 4}$

9. Given two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ with $0 < a_n \le b_n$, describe two scenarios that are inconclusive when using the Direct Comparison Test.

State the method you would use to determine the convergence of each of the following series. You do not need to determine whether the series converges or diverges.

10. $\sum_{n=1}^{\infty} \frac{6}{\sqrt[5]{n^3}}$

11. $\sum_{n=1}^{\infty} \frac{9}{5^n}$

12. $\sum_{n=1}^{\infty} \frac{2^n}{3+7^n}$

13. $\sum_{n=1}^{\infty} \frac{11n-8}{5n+7}$

Guided Notes

Limit Comparison Test Name: ________________________

Example 1: We are given the series $\sum_{n=1}^{\infty} \frac{7}{6^n - 5}$ and asked to determine whether the series converges or diverges. What methods might we try in order to answer this question?

Note: Our instincts should lead us to the Direct Comparison Test, but let's investigate the other methods we have learned as well.

1.

2.

3.

4.

None of the methods we have studied so far is applicable to this particular series.

However, there is another test we can use to determine the convergence or divergence of a series: The Limit Comparison Test.

As in the Direct Comparison Test, the Limit Comparison Test will require us to use a series that is *similar* to the series we are given.

Limit Comparison Test

Suppose that $a_n > 0$, $b_n > 0$, and $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L$ where *L* is finite and positive. Then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Let's take a look at the possibilities for *L* to see why *L* must be finite and positive in order to apply the Limit Comparison Test.

Since we must begin with $a_n > 0$ and $b_n > 0$, we know that the limit of the ratio of these terms must be nonnegative if the limit exists. Therefore, we know $L \geq 0$.

Case 1: If *L* is a ______________ value, we can conclude that the limit of the ratio of terms of ______ is this same positive finite value. Therefore when we look far enough out in the series, the terms of $\sum a_n$ are about __________________________________.

So the partial sums for $\sum a_n$ converge if and only if ____________________________.

Likewise, the partial sums for $\sum a_n$ diverge if and only if ________________________ ______________.

Case 2: If $L = 0$ or the limit does not exist, the Limit Comparison Test is inconclusive. The following examples demonstrates this.

Let's use the series $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ and $\sum_{n=1}^{\infty}\frac{1}{n}$. Both series consist of positive terms.

Now, find $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)$.

Let's change the labels and find $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)$ once again.

We know ___________________ by the ________________ (see Lesson 5) and we know that ________ is the ____________which _________. So in these examples, when __________ or _____________________, one series converges while the other diverges.

Note: There are other examples where both series converge or both series diverge even when L is not finite and positive.

To summarize, we have shown that we can only apply the Limit Comparison Test to two series consisting of positive terms if the limit of the ratio of their terms is a ___________ ___________.

Going back to **Example 1**, we can compare $\sum_{n=1}^{\infty} \frac{7}{6^n - 5}$ to ____________. When we were

trying to apply the Direct Comparison Test, we used __________as the comparison series but were unable to apply the test because ___________ has smaller terms than $\sum_{n=1}^{\infty} \frac{7}{6^n - 5}$.

Since we do not need to decide what series has larger or smaller terms in order to use the Limit Comparison Test, we will use the simpler version of the geometric series in the comparison.

The series __________ is a __________ with __________ and ___________. Therefore,

______________________.

Note that both the original series _______ and new series _________ consist of positive terms.

Let's apply the Limit Comparison Test.

Now, find $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)$.

What if we interchanged the labels before calculating the limit?

Find $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)$.

Notice that it did not matter which series we labeled as $\sum a_n$ or $\sum b_n$. Both limits produced positive, finite values.

When using the Limit Comparison Test, we can choose to label the given series as either $\sum a_n$ or $\sum b_n$. In some cases, we may find that changing how we label the series may help us simplify the limit.

However, to be consistent in these notes, we will label the given series as $\sum a_n$.

Example 2: Use the Limit Comparison Test to determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$$

Example 3: Use the Limit Comparison Test to determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{3n^2+5n-1}{4n^7+8n^3+9}$$

Example 4: Use the Limit Comparison Test to determine whether the series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{5^n}{2^n + 7}$$

Example 5: Use the Limit Comparison Test to determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{3^n}{4^n - 1}$$

Activity Sheet

Activity: Limit Comparison Test Name: ________________________

1. The Limit Comparison Test requires us to calculate $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L$. What must be true about L in order to be able to apply this test?

Determine whether each of the following series converge or diverge using the Limit Comparison Test.

2. $\sum_{n=1}^{\infty} \frac{5}{\sqrt[4]{n^3+8}}$

3. $\sum_{n=0}^{\infty} \frac{8^n + 4}{13^n + 7}$

4. $\sum_{n=1}^{\infty} \frac{8n^2 + 3n - 7}{9n^3 - 2n}$

5. When asked to determine if a series converges or diverges, what would lead you to try the Limit Comparison Test rather than other methods?

Let's assume $a_n > 0$ and $b_n > 0$. You have done the work to calculate the following limits. If you arrived at each of the following conclusions, would you be able to apply the Limit Comparison Test? Why or why not?

6. $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \infty$

7. $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{1}{3}$

8. $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = 1$

9. $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = 0$

10. $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \sqrt{7}$

Guided Notes

Ratio Test Name: ________________________

Example 1: We are given the series $\sum_{n=1}^{\infty} \frac{n}{5^n}$ and asked to determine whether the series converges or diverges. What methods might we try in order to answer this question?

1.

2.

3.

4.

None of the methods we have studied so far is applicable to this particular series.

However, there is another test we can use to determine the convergence or divergence of a series: The Ratio Test.

The Ratio Test is useful since we do not need to compare the given series to another series. We only need to use the given series and the expression for its terms. We can also use the Ratio Test for series with negative terms.

Ratio Test

Let $\sum a_n$ be a series with nonzero terms.

4. The series $\sum a_n$ converges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$.
5. The series $\sum a_n$ diverges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\infty$.
6. The Ratio Test is inconclusive if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ does not exist.

The absolute values are important, especially for series with some positive terms and some negative terms, but where the positive and negative values do not alternate from one term to the next. In a case such as this, the limit of the ratio of successive terms with the absolute value would exist, but the limit without the absolute value would not exist. We will not investigate a case such as this in the notes but interested students can ask to see an example later.

The Ratio Test requires us to find $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$. Unlike the Limit Comparison Test, we cannot calculate the limit of the reciprocal and arrive at the same conclusion. For example, if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 5$, then $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \frac{1}{5}$. This would lead to two different conclusions using the Ratio Test.

Let's apply the Ratio Test to the series in **Example 1**: $\sum_{n=1}^{\infty}\frac{n}{5^n}$.

Example 2: Use the Ratio Test to determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n$$

Example 3: Use the Ratio Test to determine whether the series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{8^n}{n^6}$$

Example 4: Use the Ratio Test to determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{6}{n}$$

Example 5: Use the Ratio Test to determine whether the series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{3}{n^4+1}$$

Example 6: Use the Ratio Test to determine whether the series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{9^n}{n!}$$

Example 7: Use the Ratio Test to determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{(n-1)!}{n^7}$$

Example 8: Use the Ratio Test to determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} n\left(\frac{-9}{7}\right)^n$$

Activity Sheet

Activity: Ratio Test Name: ________________________

Complete the following statements regarding the Ratio Test.

1. The series $\sum a_n$ must have ___________ terms.

2. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$, then $\sum a_n$ ___________.

3. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\infty$, then $\sum a_n$ ___________.

4. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=1$, ____________________________________.

5. Describe an advantage of using the Ratio Test instead of the Direct or Limit Comparison Tests.

6. Simplify the following factorial expression: $\frac{n!}{(n-5)!}$.

7. Simplify the following factorial expression: $\frac{(4n-3)!}{(4n+2)!}$.

Determine whether each of the following series converges or diverges using the Ratio Test.

8. $\sum_{n=1}^{\infty} n(n!)$

9. $\sum_{n=1}^{\infty} \frac{n^2}{(3n)!}$

10. $\sum_{n=3}^{\infty} \frac{2n^2+3n}{4^n}$

11. $\sum_{n=3}^{\infty}(n-2)\left(\frac{7}{5}\right)^{n}$

For each of the following, match the series with the method you would use to determine convergence. Some series can be determined using more than one method. However, you may list each method only once.

12. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[8]{n^7}}$

13. $\sum_{n=1}^{\infty} \frac{(n+2)!}{11}$

14. $\sum_{n=1}^{\infty} \frac{8^n}{13^n - 9}$

15. $\sum_{n=1}^{\infty} \frac{4}{3^n}$

16. $\sum_{n=1}^{\infty} \frac{9n-2}{7n+1}$

17. $\sum_{n=1}^{\infty} \frac{6}{n^2+1}$

A. The n^{th}-Term Test for Divergence

B. Geometric series

C. *p*-series

D. Direct Comparison Test

E. Limit Comparison Test

F. Ratio Test

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