Summer 2019

ALGEBRAIC TOPICS IN THE CLASSROOM – GAUSS AND BEYOND

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An Essay Submitted to the
Office of Graduate Studies
College of Arts & Science of
John Carroll University
in Partial Fulfillment of the Requirement
for the Degree of
Master of Arts

By
Lisa Krance
2019

The essay of Lisa Krance is hereby accepted:

Advisor – Patrick Chen   Date

I certify that this is an original document

Author – Lisa Krance   Date

Algebraic Topics in the Classroom - Gauss and Beyond:

I. Introduction

Imagine challenging a young student of six or seven years old to add up the integers from 1 to 100. Then imagine the utter shock when a student comes up with the correct answer in just a few minutes! Has the student actually completed the addition longhand, without the aid of a calculator? Or has the student surmised some type of formula that will efficiently provide the sum? In the late 1700's, the latter is exactly what Carl Friedrich Gauss accomplished. As a young child, to the astonishment of his teacher, Gauss "paired up" the first and the last numbers in the sequence and created the following formula that added the numbers on his slate as follows:

$$\frac{1}{2}(100)(1+100) = 50 * 101 = 5050$$

As an educator of high school students, I would be astounded if a student was able to figure out an algorithm or method of adding these numbers without prior knowledge or prompting of any kind. As a mathematics educator I work with students that are not child 'prodigies' like Gauss and I continually strive to spark student interest in mathematics and help students systematically develop strong problem solving skills that can be utilized in a variety of motivating applications. I believe that Gauss can be a tremendous mathematical role-model for students and that incorporating some of the history and background of mathematics into my classroom will significantly enrich student learning.

Johann Carl Friedrich Gauss, 1777 - 1855, is one of the greatest mathematical and scientific minds the world has ever known. Gauss started out in Braunschweig, Germany as a child prodigy and made significant contributions to numerous fields in mathematics and science until his death in Göttingen, Germany. His accomplishments are vast and include, but are not limited to, advancements in the fields of algebra, analysis, astronomy, differential geometry, matrix theory, number theory, statistics, physics, mechanics, magnetic fields, and optics.

As stated by Burton, Gauss's 1801, *'Disquisitiones Arithmeticae'* which is Latin for Arithmetical Investigations, made substantial contributions to number theory. Some of the highlights of his book include: demonstrating that a regular heptadecagon can be constructed with straightedge and compass, introducing the symbol ≅ for congruence and using it in explaining modular arithmetic, presenting the first two proofs of the law of quadratic reciprocity, the proof of the fundamental theorem of algebra and development of the theories of binary and ternary quadratic forms. Gauss also proved the following theorems: Descartes's rule of signs, Fermat's last theorem for $n = 5$, Fermat's polygonal number theorem for $n = 3$, and the Kepler conjecture for regular arrangements. Gauss has several items named in his honor such as the 'The Carl Friedrich Gauss Prize for Applications of Mathematics' and the 'gauss', G or Gs, which is a CGS unit of measurement for a magnetic field.

Gauss' accomplishments are varied and far too numerous to all be mentioned in this paper. I have selected several of his achievements related to Algebra as the focus of this paper. Great mathematical accomplishments are rarely achieved in isolation. The ideas of others must be embraced and incorporated into new applications. One of Gauss' most notable quotes was "Mathematicians stand on each other's shoulders". Gauss built upon the concepts of many mathematical giants and others have built upon Gauss' contributions. In addition to Gauss' ideas, I will present additional algebraic topics developed by other significant mathematicians that complement the topics related to Gauss. All of the topics presented are designed to ignite and enhance student learning at the high school level.

II. Sum of an Arithmetic Sequence

a. Mathematical Background of the Sum of an Arithmetic Sequence Formula

An arithmetic sequence is a list of patterned numbers where each term can be determined by adding a positive or negative number, known as a common difference $d,$ to the previous term in the sequence. If a_1 is used to denote the first term in a sequence and a_n is used to denote the n^{th} term, then terms can be determined in the following manner:

$$a_{2=} a_1 + d$$

$$a_{3=} a_2 + d = (a_1 + d) + d = a_1 + 2d$$

$$a_{4=} a_3 + d = (a_1 + 2d) + d = a_1 + 3d$$

So, in general, a_n in the arithmetic sequence can be determined using the explicit and recursive formulas stated below. The explicit formula requires knowing the first term, term number and the common difference. The recursive formula requires knowing the previous term and the common difference. The explicit formula is often more useful, especially when the previous term is unknown and you are looking to find a term that is found much later in the sequence, such as the 200$^{\text{th}}$ term, a_{200}.

Explicit formula: $a_{n=} a_1 + d(n-1)$

Recursive formula: $a_{n=} a_{n-1} + d$

The sum of the first n terms of an arithmetic sequence, S_n, is given by the following formula:

$$S_n = \frac{1}{2}n(a_1 + a_n)$$

As Gray states, a very young Gauss of age seven essentially surmised this formula on his own and utilized this method to add up the integers from 1 to 100. Determining sums in this manner can be useful in a variety of real-world applications. I will mention some of these applications in the lesson story problems.

b. Lesson on the Sum of an Arithmetic Sequence

This lesson is for students that are already familiar with arithmetic sequences and their explicit and recursive formulas. Students will also be familiar with other types of sequences, such as a geometric sequence. In addition, students will have a prior understanding of the Sigma, or summation notation. I will explain the story of Gauss and then generate the sum of an arithmetic sequence formula as follows. The sum, S_n, of the first n terms of an arithmetic sequence is given by the following:

$$S_n = \sum_{k=1}^{n} a_k$$

$$= \sum_{k=1}^{n} (a_1 + (k-1)\,d)$$

$$= na_1 + d\sum_{k=1}^{n} (k-1)$$

$$= na_1 + d\sum_{k=2}^{n} (k-1)$$

$$= na_1 + d\sum_{k=1}^{n-1} k$$

Then using the sum identity,

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

The result is:

$$S_n = na_1 + \frac{1}{2}dn(n-1) = \frac{1}{2}n[2a_1 + d(n-1)]$$

Then by the following substitution, the final summation formula for S_n can be determined:

$$a_1 + a_n = a_1 + [a_1 + d(n-1)] = 2a_1 + d(n-1)$$

$$S_n = \frac{1}{2}n(a_1 + a_n)$$

I will work through the following lesson examples for the students. For the following problems, find the sum of the arithmetic sequence, S_n using the appropriate formula(s).

1. $\left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots, \frac{15}{2}\right\}$

Steps:

a. First we will observe that this is an arithmetic sequence by determining that the numbers in the sequence are each one greater than the previous term so the common difference, d, in this sequence is one. This distinguishes the sequence from a different type, such as a geometric sequence, and we will then be able to use the formula for the sum of an arithmetic sequence.

b. Determine the number of terms in the sequence, 'n', by subtracting the first term from the last term and dividing the result by the common difference and adding one to that result. So we substitute the values in our problem below:

$$n = \frac{a_n - a_1}{d} + 1. \quad n = \frac{\frac{15}{2} - \frac{1}{2}}{1} + 1 = 7 + 1 = 8.$$

c. Substitute the values of $a_{1,}$ $a_{\mathrm{n}},$ n into the S_n formula and compute S_8.

$$S_8 = \frac{1}{2}(8)\left(\frac{1}{2} + \frac{15}{2}\right). \; S_8 = 32.$$

2. $\sum_{n=1}^{50} 2n$

Steps:

a. The explicit formula for the sequence is: $a_n = 2n$. We will observe that this is an arithmetic sequence by determining that the numbers in the sequence $\{2, 4, 6, \dots, 100\}$ are each two greater than the previous term so the common difference, d, in this sequence is two.

b. Determine 'n' to be 50. This example is different from the first as it is written in Sigma notation, so 'n'can be obtained by taking the upper limit of the series, 50, and subtracting the lower limit of the series, 1, and then adding 1 to that result: $n = 50 - 1 + 1 = 50$. Or we can utilize the method from example one: $n = \frac{100-2}{2} + 1 = 49 + 1 = 50$.

c. a_1 is determined by plugging in the lower limit of 1 into the explicit formula:

$a_1 = 2(1) = 2$

d. a_n is determined by plugging in the upper limit of 50 into the explicit formula:

$a_\text{n} = 2(50) = 100$

e. Substitute the values of $a_{1,}$ a_n, n into the S_n formula and compute S_{50}.

$S_{50} = \frac{1}{2}(50)(2 + 100).$ $S_{50} = 2550.$

3. $\sum_{n=5}^{25} n$

Steps:

a. We will observe that this is an arithmetic sequence by determining that the numbers in the sequence $\{5, 6, 7, \ldots, 25\}$ are each one greater than the previous term so the common difference, d, in this sequence is one. An alternate way to express this series would be: $\sum_{n=1}^{21}(n+4)$. The explicit formula is: $a_\text{n} = a_1 + d(n-1) = 5 + n - 1 = n + 4.$

b. Determine 'n' to be 21 by taking the n^{th} term of the sequence, 25, and subtracting the first term of the sequence, 5, and dividing the result by the common difference and then adding 1 to that result: $n = \frac{25-5}{1} + 1 = 21.$ We will be finding the sum of 21 terms in the sequence.

c. a_1 can also be determined by plugging in one for n into the explicit formula:

$a_1 = 1 + 4 = 5$

d. a_{21} can also be determined by plugging in 21 for n into the explicit formula:

$a_{21} = 21 + 4 = 25$

e. Substitute the values of $a_{1,} a_{21}, n$ into the S_n formula and compute S_{21}.

$S_{21} = \frac{1}{2}(21)(5 + 25)$. $S_{21} = 315$.

The students will then work in pairs or individually to complete the following examples. A few student volunteers will present their solutions to the problems to the class. For the following problems, find the sum of the arithmetic sequences using the formulas from the lesson:

1. $\{4, 9, 14, \ldots, 44\}$; $d = 5$, $n = \frac{44-4}{5} + 1 = 9$; $S_9 = \frac{1}{2}(9)(4 + 44) = 216$.
2. $\{10, 5, 0, \ldots, -40\}$; $d = -5$, $n = \frac{-40-10}{-5} + 1 = 11$; $S_{11} = \frac{1}{2}(11)(10 - 40) = -165$.
3. $\sum_{n=1}^{75}(2n - 4)$; $d = 2$, $a_1 = 2(1) - 4 = -2$, $a_{75} = 2(75) - 4 = 146$

   $S_{75} = \frac{1}{2}(75)(-2 + 146) = 5400$.

In a subsequent lesson I will then present the next story problem example. This problem is lengthy and will bring together a number of key concepts involving sequences and series. When I present and work through the problem with the students, I would first ask them to sketch what they think the theater seating would look like. This visualization of the seating is important to understanding how the various parts of the problem should be set up. After the students draw the theater, I would sketch the theater and they can compare their drawing to mine for accuracy. I would work through part ‘a’ for the students. I would then have the students work in groups to complete parts ‘b’, ‘c’, and ‘d’. After the students have completed the problem, I would present the steps and solutions to the remaining parts.

Lesson Example:

“A 20-row theater has two aisles and three sections. The two side sections have four chairs in the first row and one more chair in each succeeding row. The middle section of the theater has ten chairs in the first row and one more chair in each succeeding row.”

a. Write the arithmetic sequence for each section. Also write the Sigma notation for the sections.

   Sequence for the sides: 4, 5, 6, 7,…, 23

   Summation/Sigma notation for the sides: $\sum_{n=1}^{20}(n+3)$

   Sequence for the middle section: 10, 11, 12, 13,…, 29

   Summation/Sigma notation for the middle section: $\sum_{n=1}^{20}(n+9)$

b. Write the arithmetic sequence and Sigma notation for the entire theater, which is all three sections.

   Sequence: 18, 21, 24, 27,…, 75

   Summation/Sigma notation: $\sum_{n=1}^{20}(3n+15)$

c. Find the number of chairs in each section and in the entire theater.

Number of chairs in each side section: $S_{20} = \frac{1}{2}(20)(4 + 23) = 270$ chairs,

$270\ chairs * 2\ side\ sections = 540\ chairs\ in\ the\ side\ sections.$

Number of chairs in the middle section: $S_{20} = \frac{1}{2}(20)(10 + 29) = 390$ chairs,

$540 + 390 = 930\ total\ chairs\ in\ the\ theater.$

As an alternative method, the total number of chairs in the entire theater can also be computed as follows using the information from part 'b' above:

$S_{20} = \frac{1}{2}(20)(18 + 75) = 930\ total\ chairs\ in\ the\ theater.$

d. Front row tickets in the theater cost \$60. After every five rows, the ticket price goes down by \$5.00. What is the total amount of money generated by a 'full house'?

Number of chairs in the first 5 rows: $S_5 = \frac{1}{2}(5)(18 + 30) = 120$ chairs. Therefore, the maximum revenue generated by these first 5 rows is $120 * \$60 = \$7200$

Number of chairs in the next 5 rows: $\frac{1}{2}(5)(33 + 45) = 195$ chairs. Therefore, the maximum revenue generated by the next 5 rows is $195 * \$55 = \$10{,}725$

Number of chairs in the next 5 rows: $\frac{1}{2}(5)(48 + 60) = 270$ chairs. Therefore, the maximum revenue generated by the next 5 rows is $270 * \$50 = \$13{,}500$

Number of chairs in the last 5 rows: $\frac{1}{2}(5)(63 + 75) = 345$ chairs. Therefore, the maximum revenue generated by the last 5 rows is $345 * \$45 = \$15{,}525$

Total amount of money generated by a full house:

$\$7200 + \$10{,}725 + \$13{,}500 + \$15{,}525 = \$\mathbf{46{,}950}$.

This is a capstone question for a unit on arithmetic sequences and series. It brings multiple concepts together and makes a connection to the arts in the real world. I believe that this is an area that would interest many of my students. Most students are familiar with a theater of some type and should be able to make a personal connection with the concepts. I think this connection helps to facilitate a deeper level of understanding for my students. Students are able to see how the concept of an arithmetic series would apply in the world. It is important that students work through these types of problems, as they can be challenging and rewarding for students as they struggle to bring all of the parts of the problem together. When students solve problems of this nature, it can instill a sense of pride and accomplishment as they check to see if each part is correct and then eventually solve the entire problem. I would then use the following version of this question on a subsequent test over sequences and series.

Test Question:

> "There are 30 rows of seats in a theater. The first row contains 10 seats. Each successive row increases by three seats. Write a series using Sigma notation for the number of seats in the theater. How many seats are in the last row? How many seats are there in the theater? If a show is a sell out and seats in the front half of the theater, the first 15 rows, cost $80 and the back half of the seats cost $60, what is the total amount of money brought in by the sold-out show?
>
> *Solution:*
>
> - $Sequence$: $10, 13, 16, \ldots, 97$; $Series$: $\sum_{n=1}^{30}(3n+7)$

- $3(30) + 7 = 97\ seats\ in\ the\ last\ row,\ S_{30} = \frac{1}{2}(30)(10 + 97) =$

  $1605\ seats\ in\ the\ entire\ theater.$
- *The number of seats in the first 15 rows:* $\frac{1}{2}(15)(10 + 52) = 465.$ *Therefore, the maximum revenue generated by the first 15 rows is* $465 * \$80 = \$37{,}200$
- *The number of seats in the last 15 rows:* $\frac{1}{2}(15)(55 + 97) = 1140.$ *Therefore, the maximum revenue generated by the last 15 rows is* $1140 * \$60 = \$68{,}400$
- $\$37{,}200 + \$68{,}400 = \$105{,}600$ for a sold out show.

III. Gaussian Elimination Method of Solving Systems of Equations

a. Mathematical Background of Gaussian Elimination

Gaussian elimination, also known as row reduction, is a method that can be used to solve systems of linear equations. According to Gracr, despite this method being named after Gauss, others contributed to the development of this method. An ancient source, '*The Jiuzhang Suanshu'*, or Nine Chapters of the Mathematical Art, is a collection of problems that was completed in the third century. Some of the problems are systems of equations that appear to have been solved using the Gaussian elimination method. Independently in Europe, mathematicians such as Issac Newton and Michel Rolle solved systems using similar substitution methods in the late 17th century. Gauss worked off of Newton's published notes and created the notation for this elimination method in 1810. Gauss utilized this elimination method in developing the method of least squares with Adrien-Marie Legendre to make statistical models for unknowns in simultaneous linear equations.

The Gaussian elimination method involves converting a system of equations into an augmented matrix and then using elementary row operations to obtain reduced row echelon form. The left hand square matrix will be the multiplicative identity matrix and the solution to the system will be in the right hand column of the matrix. For example, the solution represented by the following matrix would be $x = -2 \; and \; y = 3$.

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$$

b. Lesson on Gaussian Elimination

This lesson would ideally be used with students that are familiar with matrices, matrix operations, determinants, multiplicative and additive identities and inverses of matrices. It will also work well within a larger unit that uses inverse matrices and Cramer's rule methods to solve systems of linear equations. After explaining the background outlined above, I would solve the systems of equations in the following two problems using elementary row operations to obtain the identity matrix and the solution to the system as follows:

$$\begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \end{bmatrix} \quad or \quad \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{bmatrix}$$

Elementary row operations include: switching two rows, multiplying a row by a non-zero constant, adding one row to another, and any combination of these steps. I would stress that one strategy that will always work is to obtain a one in a column first and then use this one to create zeros for the other entries in that column. This can be done for all of the columns in a problem. However, I would explain that students can obtain the ones and zeros in any order if they happen to see a more efficient sequence to obtain the identity matrix. Students will have practice problems to work on in class, as time allows, and for homework. In a subsequent lesson, I would show students how to use the reduced row echelon form function, '*rref*', in the matrix menu of their graphing calculators to check and obtain solutions.

I will work through the following two lesson examples of systems of equations to be solved using Gaussian elimination. As the problem is worked through, I would point out that Gaussian elimination is very similar to the 'elimination method', sometimes referred to as 'linear combinations', that students are familiar with using to solve a system of linear equations. For the steps in the first example, I would point out what the corresponding similar steps would be using the elimination method. I would also point out that the substitution method is available for solving systems of equations. In addition, linear systems with two variables can be solved by graphing the two lines and finding the intersection point. These four methods provide students with a variety of options of solving systems.

1. $12x = 3y + 144$
   $16x + 4y = -64$

Steps:

a. All equations must be written in standard form, so transform the first equation:

   $12x - 3y = 144$

b. Write the system as an augmented matrix:

   $$\begin{bmatrix} 12 & -3 & 144 \\ 16 & 4 & -64 \end{bmatrix}$$

c. To obtain a one in the first row, first column, multiply the first row by $\frac{1}{12}$ to replace the first row: $\begin{bmatrix} 1 & -\frac{1}{4} & 12 \\ 16 & 4 & -64 \end{bmatrix}$

d. To obtain a zero in the second row, first column, multiply the top row by -16 and add the result to the second row to replace the second row: $\begin{bmatrix} 1 & -\frac{1}{4} & 12 \\ 0 & 8 & -256 \end{bmatrix}$

e. I would point out to students the corresponding steps using the elimination method which would include dividing the first equation by 12, and then multiplying the second equation by -16, and finally adding the two resulting equations together. This would eliminate $'x'$ and give you an equation in which you could then go on to solve for $'y'$.

f. To obtain a one in the second row, second column, multiply the second row by $\frac{1}{8}$:

$$\begin{bmatrix} 1 & -\frac{1}{4} & 12 \\ 0 & 1 & -32 \end{bmatrix}$$

g. To obtain a zero in the first row, second column, multiply the second row by $\frac{1}{4}$ and add the result to the first row to replace the first row: $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -32 \end{bmatrix}$

h. Now that the multiplicative identity has been obtained in the first two columns, the solution to the system is in the third column: $x = 4 \; and \; y = -32$.

2. $$\begin{aligned} -2x + y - z &= 2 \\ -x - 3y + z &= -10 \\ 3x + 6z &= -24 \end{aligned}$$

Steps:

a. Write the system as an augmented matrix: $\begin{bmatrix} -2 & 1 & -1 & 2 \\ -1 & -3 & 1 & -10 \\ 3 & 0 & 6 & -24 \end{bmatrix}$

b. To obtain a one in the first row, first column, multiply the second row by -1 and switch the first and second rows: $\begin{bmatrix} 1 & 3 & -1 & 10 \\ -2 & 1 & -1 & 2 \\ 3 & 0 & 6 & -24 \end{bmatrix}$

c. To obtain a zero in the second row, first column, multiply the first row by 2 and add the result to the second row to replace the second row: $\begin{bmatrix} 1 & 3 & -1 & 10 \\ 0 & 7 & -3 & 22 \\ 3 & 0 & 6 & -24 \end{bmatrix}$

d. To obtain a zero in the third row, first column, multiply the first row by -3 and add the result to the third row to replace the third row: $\begin{bmatrix} 1 & 3 & -1 & 10 \\ 0 & 7 & -3 & 22 \\ 0 & -9 & 9 & -54 \end{bmatrix}$

e. To obtain a one in the second row, second column, multiply the second row by $\frac{1}{7}$ and replace the second row: $\begin{bmatrix} 1 & 3 & -1 & 10 \\ 0 & 1 & -\frac{3}{7} & \frac{22}{7} \\ 0 & -9 & 9 & -54 \end{bmatrix}$

f. To obtain a zero in the first row, second column, multiply the second row by -3 and add the result to the first row to replace the first row: $\begin{bmatrix} 1 & 0 & \frac{2}{7} & \frac{4}{7} \\ 0 & 1 & -\frac{3}{7} & \frac{22}{7} \\ 0 & -9 & 9 & -54 \end{bmatrix}$

g. To obtain a zero in the third row, second column, multiply the second row by 9 and add the result to the third row to replace the third row: $\begin{bmatrix} 1 & 0 & \frac{2}{7} & \frac{4}{7} \\ 0 & 1 & -\frac{3}{7} & \frac{22}{7} \\ 0 & 0 & \frac{36}{7} & -\frac{180}{7} \end{bmatrix}$

h. To obtain a one in the third row, third column, multiply the third row by $\frac{7}{36}$ to replace the third row: $\begin{bmatrix} 1 & 0 & \frac{2}{7} & \frac{4}{7} \\ 0 & 1 & -\frac{3}{7} & \frac{22}{7} \\ 0 & 0 & 1 & -5 \end{bmatrix}$

i. To obtain a zero in the second row, third column, multiply the third row by $\frac{3}{7}$ and add the result to the second row to replace the second row: $\begin{bmatrix} 1 & 0 & \frac{2}{7} & \frac{4}{7} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -5 \end{bmatrix}$

j. To obtain a zero in the first row, third column, multiply the third row by $-\frac{2}{7}$ and add the result to the first row to replace the first row: $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -5 \end{bmatrix}$

k. Now that the multiplicative identity has been obtained in the first three columns, the solution to the system is in the third column: $x = 2, y = 1, and\ z = -5.$

Depending upon the length of the class period, the lesson may take one or two days to complete. The following four problems are for the students to solve independently using Gaussian elimination. Students will start in class and then complete for homework. I will have the students choose two of the four problems that they will solve in two or more ways. I will have the students explore different orders and methods of obtaining the ones and zeros. Students can also solve one or more of the problems using substitution or elimination methods and by graphing the two-variable systems. This exploration is an important part of mathematics, as students are able to use trial and error and test different ideas they have to see their efficacy. This promotes original and creative thought in mathematics. Choosing different routes to arrive at the same correct answer helps students to examine and understand problems in different ways and provides additional levels of learning. It can also promote viewing a problem from multiple perspectives and help them make critical connections to other areas of mathematics and different disciplines.

1. $2x - y = 2$
   $x + 3y = 22$

   $Solution: x = 4, y = 6$

2. $2x + 3y - z = 1$
   $-4x + 9y + 2z = 8$
   $-2x + 2z = 3$

   $Solution: x = \frac{1}{2}, y = \frac{2}{3}, and\ z = 2$

3. $x + 2y = -1$
   $2x + 5y = -4$

   $Solution: x = 3, y = -2$

4. $x + 4y - z = 4$
   $x - 2y + z = -2$
   $5x - 3y + 8z = 13$

   $Solution: x = 4, y = -2, and\ z = 13$

IV. Descartes' Rule of Signs

a. Mathematical Background of Descartes' Rule of Signs

Rene Descartes was a French mathematician and scientist that lived from 1596-1650, over a century before Gauss. Descartes is often credited as the "father of analytical geometry' and was an early contributor to infinitesimal calculus and analysis. Descartes' Rule of Signs is a commonly used methodology that describes the roots of polynomial equations. The rule allows you to determine the possible number of positive and negative real roots of a polynomial equation with real coefficients. Descartes' Rule is often used in conjunction with the Fundamental Theorem of Algebra, the Conjugate Root Theorems and other methods to find roots of polynomials and sketch graphs of polynomials.

Descartes' Rule states that if $P(x)$ is a polynomial equation with real coefficients written in standard form then:

- The number of positive real roots of $P(x) = 0$ is either equal to the number of sign changes between consecutive coefficients of $P(x)$ or is less than that by an even number.
- The number of negative real roots of $P(x) = 0$ is either equal to the number of sign changes between consecutive coefficients of $P(-x)$ or is less than that by an even number.

In both of the above cases, multiple roots are counted according to their multiplicity. For example, a polynomial equation with zeros of $-2(mult. 3)\ and\ 5$ would have four real roots.

b. Lesson on Descartes' Rule of Signs

I would incorporate Descartes' Rule of Signs in a unit on strategies that can be utilized to solve higher degree polynomial equations. After explaining the history and the theorem, I would work through the following example problems in one lesson:

1. What does Descartes' Rule of Signs tell you about the number of positive real roots and negative real roots of: $f(x) = 2x^3 + 2x^2 - 5\text{x} - 2$?

- Since there is only one sign change between consecutive coefficients of $f(x)$, there is only one positive real root. You cannot go less than one root by an even number, as that would give you a negative number of positive roots.
- $f(-x) = -2x^3 + 2x^2 + 5\text{x} - 2$ has two sign changes between consecutive coefficients; therefore there are either zero or two negative real roots.
- Using the rational root theorem, the possible rational roots of $f(x)$ are $\pm 2, \pm 1, \pm \frac{1}{2}$. Then using synthetic division, one negative root is found to be -2. The quadratic formula can be used to solve the equation that is obtained from the division, $2x^2 - 2\text{x} - 1 = 0$, the other negative root is $\frac{1}{2}(1 - \sqrt{3})$ and the one positive root is $\frac{1}{2}(1 + \sqrt{3})$.

2. What does Descartes' Rule of Signs tell you about the real roots of:

   $h(x) = 2x^3 + x^2 - 9$?

   - Since there is one sign change between consecutive coefficients of $h(x)$, there is one positive real root. $h(-x) = -2x^3 + x^2 - 9$ has two sign changes between consecutive coefficients; therefore there are either zero or two negative real roots.
   - Then using the rational root theorem, the possible rational roots are $\pm 9, \pm 3, \pm 1, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{9}{2}$. Using synthetic division, one positive root is found to be $\frac{3}{2}$. The equation that you obtain from the division, $x^2 + 2\text{x} + 3 = 0$, has two complex roots. So there are no negative roots.

The following are lesson problems for students to solve individually or in a small group. The solutions will then be reviewed with the students the next day that the class meets.

1. What does Descartes' Rule of Signs tell you about the number of positive real roots and negative real roots of: $g(x) = 2x^4 - x^3 + x^2 - \text{x} - 1$?

- Since there are three sign changes between consecutive coefficients of $g(x)$, there are either one or three positive real roots. $g(-x) = 2x^4 + x^3 + x^2 + \text{x} - 1$ has one sign change; therefore there is one negative real root. The possible rational roots are $\pm 1, \pm \frac{1}{2}$. The one negative root is found to be $-\frac{1}{2}$. The equation that you obtain from division, $x^3 - x^2 + \text{x} - 1$, can be factored into $(x^2 + 1)(x - 1)$. So the positive root is 1 and $(x^2 + 1)$ gives two complex roots of $\pm i$.

2. What does Descartes' Rule of Signs tell you about the real roots of $k(x) = 3x^4 - 17x^3 + 27x^2 - 7\text{x} - 6$?

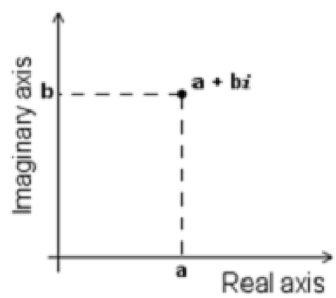
- Since there are three sign changes between consecutive coefficients, there are either three or one positive real roots. $k(-x) = 3x^4 + 17x^3 + 27x^2 + 7\text{x} - 6$ has only one sign change; therefore, there is one negative real root. The possible rational roots are $\pm 6,\ \pm 3,\ \pm 2,\ \pm 1,\ \pm \frac{1}{3},\ \pm \frac{2}{3}$. All of the roots are rational. There is one negative and three positive roots, $x = -\frac{1}{3}, 1, 2\,, 3$.

V. Complex Numbers

a. Mathematical Background of Complex Numbers

Gauss made contributions to nearly all fields of mathematics. Gauss is quoted as saying "mathematics is the queen of the sciences, and the theory of numbers is the queen of mathematics". Number theory is often cited as his favorite area of mathematics. Per Gray, in the early 1800's Gauss is credited with providing the first clear and thorough explanation of complex numbers. A number is called a complex number if it is of the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$. When b is nonzero, bi is called a pure imaginary number. Pure imaginary numbers had been in use since as early as the 16th Century to solve equations and Leonhard Euler utilized imaginary numbers in unique ways during the 18th Century. During the 19th Century, Gauss was the first to popularize the graphical interpretation of complex numbers on the complex number plane and he introduced the standard notation of '$a + bi$'. A complex number, $a + bi$, where a and b are real numbers, can be represented graphically as the point (a, b) using the real axis and the imaginary axis. Pure imaginary and real numbers together form the complex plane. Diagram 1 elaborates on how complex numbers are depicted on the complex number plane.

Diagram 1



b. Lesson on Complex Numbers

Complex numbers are typically introduced to students in a standard Algebra 2 course. When students solve quadratic equations in previous courses, complex number solutions are avoided. I first would introduce complex number solutions in an Algebra 2 course when solving quadratic equations.

I would first explain the definition of an imaginary number as follows:

- A pure imaginary number, bi, is obtained by taking the square root of a negative real number. An odd root of a real negative number will just result in a negative real number.
- The pure imaginary number, i, is defined as a number whose square is -1: $i^2 = -1$.
- Standard form of a complex number is '$a + bi$', where the real number a is called the real part and the real number b is called the imaginary part of the complex number. The complex number $a + bi$ can be identified by the point (a, b) on the complex number plane. In addition,

  $a + bi = 0$

  $\Rightarrow (a + bi)(a - bi) = 0$

  Since $(a + bi)(a - bi) = a^2 - abi + abi - b^2i^2 = a^2 - b^2i^2 = a^2 + b^2$

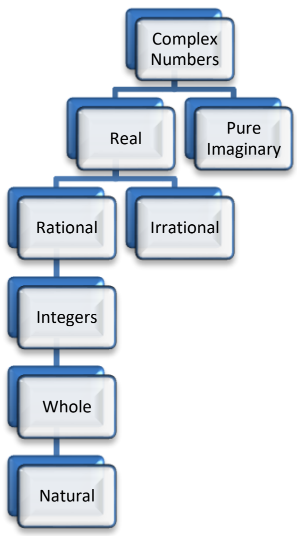
  $\Rightarrow a = 0$ and $b = 0$. This shows that the complex number 0 is identified by the origin $(0,0)$.

  It follows that $a + bi = c + di \Leftrightarrow a = c$ and $b = d$

  This shows that the identification of complex numbers by points in a plane is a one-to-one correspondence.

I would explain a brief history of complex numbers as outlined in the history section. I would then review Chart 1 for the students that explains the relation amongst number sets. I would give detail on each number set. I would explain that the broadest classification of numbers that we are concerned with is the complex numbers and that a complex number may be a real number, a pure imaginary number, or the sum of them. A pure imaginary number is of the form ai, where $a \neq 0$, such as the numbers: $6i$ or $-7i$. Students are already familiar with the set of real numbers and the subsets of real numbers. I would review rational versus irrational numbers and then the subsets of rational numbers for a complete view of complex numbers.

Chart 1



The initial problems that I would work through in the lesson would involve simplifying complex numbers. Students would be familiar with simplifying radicals with positive radicands and writing the solution in standard form. Examples to simplify complex numbers:

1. $\sqrt{-121} = 11i$, which is a pure imaginary number
2. $10 + \sqrt{-72} = 10 + \sqrt{-1 * 36 * 2} = 10 + 6i\sqrt{2}$
3. $6 - 3\sqrt{-81} = 6 - 3(9i) = 6 - 27i$
4. $\sqrt{-144} - 2\sqrt{25} = 12i - 2(5) = 12i - 10 = -10 + 12i$

We would then solve the following quadratic equations to obtain imaginary solutions. We would use the square root method to solve the first example and the quadratic formula to solve the second example.

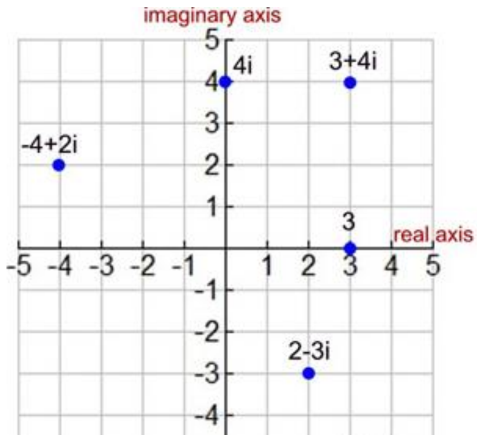
1. $2x^2 + 22 = 4, \quad x^2 = -9, x = \sqrt{-9}, \; x = \pm 3i$

2. $x^2 - 2x + 2 = 0$ Using the quadratic formula:

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

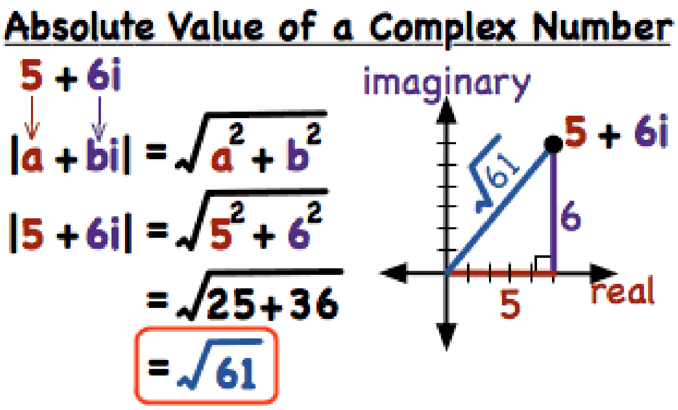
The lesson would also include graphing numbers on the complex plane. In the complex number plane, the real axis is horizontal and the imaginary axis is vertical. We would graph the complex number plane and graph the following numbers: $3, 4i, 3 + 4i, 2 - 3i, and - 4 + 2i$ as shown in diagram 2.

Diagram 2



The final part of the lesson would include finding the absolute value of complex numbers. We would first plot the complex number, such as $-4i$. Then I would explain that the geometric meaning of the absolute value of a number is the distance from the point in the plane to the origin. I would illustrate that $-4i$ is 4 units away from the origin, so $|-4i| = 4$. Then we would find the absolute value of $5 + 6i$. Diagram 3 shows how we would plot $5 + 6i$ on the complex number plane and then illustrates the steps that utilize the Pythagorean Theorem to determine that $|5 + 6i| = \sqrt{61}$.

Diagram 3



Below are the lesson problems for students to work through on their own or in a small group. The solutions will be reviewed step-by-step with the students at the end of the lesson.

1. Simplify: $\sqrt{-300}$. *Solution*: $10i\sqrt{3}$
2. Solve: $x^2 - 20 = 2x^2 + 4$. *Solution*: $x = \pm 2i\sqrt{6}$
3. Solve: $x^2 + 2x + 3 = 0$. *Solution*: $x = -1 \pm i\sqrt{2}$
4. Find $|4 - 2i|$. *Solution*: $2\sqrt{5}$

c. Lesson Extension on Complex Numbers

As a supplement to the basics with complex numbers, I would teach students to perform operations with complex numbers: addition, subtraction, multiplication and division. I will also work through the patterns associated with i. Addition and subtraction can simply be explained by adding/subtracting the real parts of the numbers and adding/subtracting the imaginary parts of the numbers. The following are examples to work through for students:

1. $(3+i)+(8+5i)=(3+8)+(i+5i)=11+6i.$
2. $(-9-4i)-(3-7i)=-9-4i-3+7i=-12+3i.$

Multiplication and division are slightly more complicated. A key idea that I would remind students of is that i^2 is the real number -1. I would also stress that addition, subtraction, multiplication and division of complex numbers can result in a real number adding to a pure imaginary number answer. Examples to work through for students:

1. $(2i)^2-5i^2=4i^2-5i^2=\ -1i^2=(-1)(-1)=1$
2. $(3-6i)(1+2i)=3+6i-6i-12i^2=3-12(-1)=15$
3. $(-2+4i)^2=4-16i+16i^2=4-16i-16=-12-16i$
4. Simplify: $\frac{5}{2i}$ Multiply by $\frac{i}{i}$: $\frac{5}{2i}*\frac{i}{i}=\frac{5i}{2i^2}=\frac{5i}{2(-1)}=-\frac{5}{2}i$
5. Simplify: $\frac{1-2i}{3+i}$ Multiply by the complex conjugate of the denominator $\frac{3-i}{3-i}$:

$$\frac{1-2i}{3+i}*\frac{3-i}{3-i}=\frac{3-i-6i+2i^2}{9-i^2}=\frac{3-7i-2}{9—1}=\frac{1-7i}{10}=\frac{1}{10}-\frac{7}{10}i$$

I would also include an explanation of the “pattern” associated with i. Since we have worked with i and i^2, and perhaps come across i^3 in a multiplication problem, I would explain that we can easily determine the value of i raised up to any whole number power. I would present Table 1 and work through the methodology that explains that i raised up to any whole number exponent will result in one of the four values: $1, -1, -i, or\ i$.

Table 1

$i = \sqrt{-1}$	by definition
$i^2 = -1$	by definition
$i^3 = -i$	$i^3 = i(i^2) = i(-1) = -i$
$i^4 = 1$	$i^4 = i^2(i^2) = (-1)(-1) = 1$
$i^5 = i$	$i^5 = i^4(i) = 1(i) = i$
$i^6 = -1$	$i^6 = i^4(i^2) = 1(-1) = -1$
The pattern will continue for all subsequent powers of i.	

I would then work through the following examples for the students explaining that the pattern consists of four answer options, so the power will be divided by four. Then, like in modular arithmetic, the remainder will provide the answer to the problem.

1. Find i^{45}

Steps:

a. Divide: $45 \div 4$. The remainder will tell you whether the answer is $1, -1, -i, or\ i$ as outlined in Table 2.

Table 2

Remainder	**Solution**
1	i
2	-1
3	$-i$
0 *(no remainder)*	1

b. The reminder on this problem is one. So by looking at Table 2, $i^{45} = i$.

2. Find i^{971}

Steps:

a. Divide: $971 \div 4$.

b. The reminder on this problem is three. So, $i^{971} = -i$.

Below are problems for students to work through on their own or in a small group that review the concepts that were taught in the lesson extension on complex numbers.

1.	Simplify: $(2+4i)-(3+2i)$.	*Solution*: $-1+2i$
2.	Simplify: $(-6-5i)(1+3i)$.	*Solution*: $9-23i$
3.	Simplify: $(4i)^2+3i^2$.	*Solution*: -19
4.	Simplify: $\frac{18}{3i}$	*Solution*: $-6i$
5.	Simplify: $\frac{i+2}{i-2}$	*Solution*: $-\frac{3}{5}-\frac{4}{5}i$
6.	Simplify: i^{109}	*Solution:* $Remainder\ is\ 1, so\ i^{109} = i$

VI. The Fundamental Theorem of Algebra

a. Mathematical Background of The Fundamental Theorem of Algebra

In 1799 Gauss provided a new proof of the theorem that every polynomial function of one variable with rational coefficients can be resolved into real factors of the first or second degree, thus proving the fundamental theorem of algebra. Gauss was 22 years old. The fundamental theorem of algebra states that every non-constant single-variable polynomial equation with complex coefficients has at least one complex root. Corollaries of the theorem state that every polynomial equation of degree $n \geq 1$ has exactly n roots, including multiple roots and complex roots and there will be n linear factors.

As Struik asserts, mathematicians including Jean le Rond d'Alembert had produced false proofs before him, and Gauss's dissertation contains a critique of Jean le Rond d'Alembert's work. Using generally accepted standards of today, Gauss's own attempt is not quite acceptable, due to the implicit use of the Jordan curve theorem. However, he subsequently produced three other proofs, the last one in 1849 being generally rigorous. His attempts clarified the concept of complex numbers considerably along the way.

I would point out to students that they have typically had courses in pre-algebra, algebra 1 and approximately half of an algebra 2 course before they are formally presented with the *fundamental* theorem of algebra. I mention this to students because they commonly think that they have learned the majority of algebraic topics in an algebra 1 course, but they really have learned the basics of algebra. Students have much more to learn in this complex subject that is the basis for higher levels of mathematics, such as calculus. This is generally because 'lower level' algebra courses focus on mastery of linear and quadratic polynomial equations and an introduction to exponential and radical equations. This theorem is a critical idea in algebra as it provides the basis for factoring, solving and graphing any higher degree polynomial equation.

b. Lesson on The Fundamental Theorem of Algebra

The fundamental theorem of algebra would be an integral part of a lesson on higher degree polynomials. Students will already have the skills of performing addition, subtraction, multiplication and division of polynomials. Students will also know how to graph polynomials, including finding zeros as x-intercepts and how to solve quadratic equations using various methods. Students would also be familiar with the rational root theorem and the conjugate root theorems. The conjugate root theorems can be referred to as the irrational root theorem and the imaginary root theorem.

According to Charles et al., the irrational root theorem states that if $P(x)$ is a polynomial equation with rational coefficients, then irrational roots of $P(x) = 0$ that have the form $a + \sqrt{b}$, with a and b rational, occur in conjugate pairs. Meaning if $a + \sqrt{b}$ is an irrational root of $P(x) = 0$ with a and b rational, then the conjugate $a - \sqrt{b}$ will also be a root. The imaginary root theorem states that if $P(x)$ is a polynomial equation with real coefficients, then the complex conjugate roots of $P(x) = 0$ occur in conjugate pairs. Meaning if $a + bi$ is a complex root of $P(x) = 0$ with a and b real, then the conjugate $a - bi$ will also be a root. After providing this review of these root theorems and background of the Fundamental Theorem of Algebra, I would work through the following examples for students. The example problems tie the Fundamental Theorem of Algebra and all of the root theorems together nicely. The examples show how the theorems are related and then used together to solve problems.

Lesson Examples: Find all the roots of the polynomial equations (rational, irrational, & imaginary solutions). Provide exact values. Then write the polynomial in linear factored form.

1. $x^3 - 2x^2 - 17x + 28 = 0$

   Steps:

   a. I would review the rational root theorem which states that all of the possible rational roots of a polynomial equation with integer coefficients are limited to the $\frac{factors\ of\ the\ constant}{factors\ of\ the\ leading\ coefficient}$, as stated by Charles, et al. So the possible rational roots are for the equation include: $\pm 28, \pm 14, \pm 7, \pm 4, \pm 2, \pm 1$. Those 12 possible rational roots could be checked to determine which will provide a zero remainder using synthetic or long division. The actual rational root(s) can also be found by graphing to find the zeros using option 2 of the *Calc* menu on the graphing calculator. Performing synthetic division on -4 would provide a zero remainder, therefore that is an actual rational root. In addition, I would show students how to enter the 12 possible rational roots into the table of the graphing calculator. We would see that only an $x - value$ of -4 would provide a $y - value$ of zero.

c. Using division, we would find that we now have to solve the quadratic equation $x^2 - 6x + 7 = 0$. Using the quadratic formula or completing the square we would find the additional irrational roots of $3 + \sqrt{2}$, $3 - \sqrt{2}$.

$$x = \frac{6 \pm \sqrt{(6)^2 - 4(1)(7)}}{2(1)} = \frac{6 \pm \sqrt{8}}{2} = \frac{6 \pm 2\sqrt{2}}{2} = 3 \pm \sqrt{2}$$

d. The equation has three roots, as the fundamental theorem of algebra assures that a cubic polynomial would have: $-4, 3 + \sqrt{2}$, and $3 - \sqrt{2}$.

e. Linear factors: $(x + 4)\left(x - \left(3 + \sqrt{2}\right)\right)\left(x - \left(3 - \sqrt{2}\right)\right)$

2. $2x^4 + 5x^3 + 15x^2 + 45x = 27$

Steps:

a. We would set the equation equal to zero and write in standard form as follows:

$2x^4 + 5x^3 + 15x^2 + 45\text{x} - 27 = 0$

b. The possible rational roots are: $\pm 27, \pm 9, \pm 3, \pm 1, \pm \frac{27}{2}, \pm \frac{9}{2}, \pm \frac{3}{2}, \pm \frac{1}{2}$. We would then determine the rational root(s) by graphing or by entering the 16 possible rational roots into the table of the graphing calculator to find the zeros. We would see that $x - values$ of -3 and $\frac{1}{2}$ provide a $y - value$ of zero. Therefore, -3 and $\frac{1}{2}$ are rational roots.

c. Using synthetic division on -3 first and then on $\frac{1}{2}$ using the new numbers obtained from the division by -3, we would find that we now have to solve the quadratic equation $x^2 + 9 = 0$. Solving this equation would provide the additional roots of $3i$ and $-3i$.

d. The equation has four roots, as the fundamental theorem of algebra assures that a quartic polynomial would have: $-3, \frac{1}{2,}, 3i, -3i$.

e. Linear factors: $(x + 3)(2x - 1)(x - 3i)(x + 3i)$

3. Write a polynomial with rational coefficients that has roots of $\sqrt{3}, 4i, -1$ in factored form and then in standard form. What is the least degree of the polynomial?

Steps:

a. Using the conjugate roots theorems, we would list the solutions that were not provided, $-\sqrt{3}$ and $-4i$, and write out the factors of the polynomial, including the conjugate roots:

$(x + 1)(x + \sqrt{3})(x - \sqrt{3})(x - 4i)(x + 4i)$

b. Multiply the two sets of conjugate factors:

$(x + 1)(x^2 - 3)(x^2 + 16)$

c. Multiply the quadratic factors:

$(x + 1)(x^4 + 13x^2 - 48)$

d. The final multiplication of the linear binomial and the quartic trinomial provides the standard form of a 5th degree, quintic polynomial:

$x^5 + x^4 + 13x^3 + 13x^2 - 48x - 48$

The following are problems for students to work through on their own or in a small group during class. These two problems provide the students with the opportunity to utilize all of the root theorems and the fundamental theorem of algebra. The problems are essentially opposites. The first problem provides the roots to a polynomial equation and then asks students to write the polynomial in factored and standard forms. Knowledge of the conjugate root theorems is needed to start the problem and write all of the factors. Then after algebraic manipulation, the fundamental theorem of algebra is displayed as the students see that the four roots provide a fourth-degree polynomial. The second problem asks students to find the roots of a quartic equation and then write the linear factors from the roots. Knowledge of the root theorems is required and the fundamental theorem of algebra is reinforced. Together the problems require students to work through polynomial problems 'backwards and forwards' as on one you are given the roots and have to write the equation and on the other problem you have to find the roots from the given equation. The problems provide for a complete knowledge and use of the root theorems and the fundamental theorem of algebra and allow students to see how problems can be created and solved. I would include these two types of problems on a higher-order polynomial test or quiz assessment.

1. Write a polynomial with rational coefficients with roots of $\sqrt{5}, 7i$ in factored form and then in standard form. What is the least degree of the polynomial?

Solution: Factored form: $(x - \sqrt{5})(x + \sqrt{5})(x - 7i)(x + 7i)$

$$= (x^2 - 5)(x^2 + 49)$$

Standard form: $x^4 + 44x^2 - 245$. The resulting polynomial has four solutions, four linear factors and it is a classified by degree as a quartic.

2. Find all the roots of the polynomial equation $6x^4 - 13x^3 + 13x^2 - 39x - 15 = 0$. Then write the linear factors of the polynomial.

   Solution: The possible rational roots include:

   $\pm 15, \pm 5, \pm 3, \pm 1, \pm\frac{5}{2}, \pm\frac{5}{6}, \pm\frac{1}{2}, \pm\frac{1}{6}, \pm\frac{5}{3}, \pm\frac{1}{3}, \pm\frac{15}{2}, \pm\frac{3}{2}, \pm\frac{1}{2}$. The actual rational roots are $-\frac{1}{3}$ *and* $\frac{5}{2}$. Performing synthetic division on the two rational roots and then solving the resulting equation, $x^2 + 3 = 0$, would provide the additional complex roots of $i\sqrt{3}$ *and* $-i\sqrt{3}$. The linear factors of the polynomial:

   $(3x + 1)(2x - 5)\left(x - i\sqrt{3}\right)\left(x - i\sqrt{3}\right)$

VII. Fixed Point Iteration Method of Approximating Roots

a. Mathematical Background of the Fixed Point Iteration Method

Fixed point iteration is a numerical method that can be utilized to solve certain equations. The fixed point of a function is an element of the function's domain that is mapped to itself by the function. The method utilizes a root-finding algorithm to find successively better approximations for roots of equations. Given a function f whose range is contained in its domain and a given point, x_0, in the domain of the function, the fixed point iteration is:

$x_{n+1} = f(x_n), n = 0,1,2, \ldots$ which gives a sequence $x_0,\ x_1,\ x_2, \ldots$

If this sequence converges to a point x in its domain, then x is a fixed point of the function; provided that the function is continuous at x. By definition, c is a fixed point of the function f if $f(c) = c$. Certain functions have fixed points and others do not. A function that has a fixed point 'c' also means that the point $\big(c, f(c)\big)$ is on the line $y = x$ so that the graph of f has a point in common with that line. For completeness, we will state and prove the Fixed Point Iteration Theorem in the following.

The theorem states: Suppose that $f: D \to D$ and $x_0 \in D$.

Put $x_1 = f(x_0),\ x_2 = f(x_1) = f\big(f(x_0)\big) = f^2(x_0), \ldots$

$x_{n+1} = f(x_n) = f(f(\ldots f(x_0)))\ (n + 1\ times) = f^{n+1}(x_0)\ (n \in \mathbb{N} \cup \{0\})$

Suppose $\lim_{n\to\infty} x_n = c, c \in D$, and f is continuous at c. Then $f(c) = c$; i.e. c is a fixed point of f.

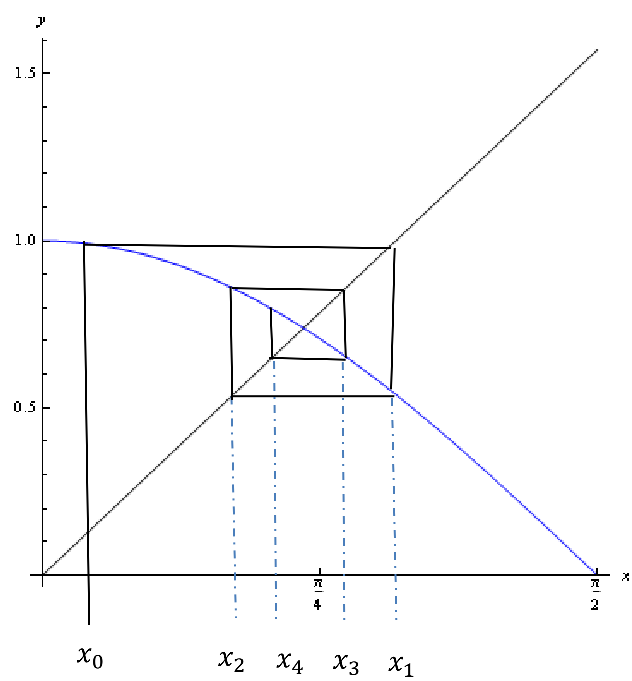
Proof: Since (x_n) is in D and $\lim_{n\to\infty} x_n = c \ \Rightarrow\ \lim_{n\to\infty} f(x_n) = f(c),$ by the continuity of f at c. Since the sequence $\{\, x_1,\ x_2,\ x_{3,}\ x_4, \ldots\}$ converges to c, the sequence $\{\, x_2,\ x_{3,}\ x_4, \ldots\}$ also converges to c. It follows that $f(c) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = c.$

The method can be most useful if traditional algebraic methods do not exist to find the roots of a certain equation. It is useful to have a wide array of techniques to find roots of equations. Depending upon the equation type and complexity, it is often convenient to use a certain method over another. Alternative methods also provide different ways of looking at a problem, different ways of thinking, and varying perspectives. Each method has its own positive points and shortcomings. The drawback to the fixed point method is that it is somewhat limited in its use, as it cannot obtain roots for certain equations.

b. Lesson on the Fixed Point Iteration Method

The Fixed Point Iteration Method can be introduced to students in a Pre-Calculus course. It can be incorporated into a lesson on solving trigonometric and algebraic equations. I would provide a brief explanation of the method and would work through the example of $x = cosx$. Diagram 4 contains two graphs where the line on the graph represents $y = x$ and the curved graph represents $y = cosx$. In Diagram 4, we can clearly see the intersection of the graphs, so there is at least one solution on the interval $\left[0, \frac{\pi}{2}\right]$. The diagram also depicts how the fixed point method is used to spiral towards the root.

Diagram 4



Since there are no algebraic methods that will solve this equation, I would explain that the fixed point method can be used to approximate the roots. I would start with an initial estimate of a solution of x, say $x_0 = 0.20$. Numerically, I will then show the following process on the graphing calculator. You enter $cosx$ into y_1 of the graphing screen, as if you were going to graph the equation. Then back on the calculation screen, enter the initial guess, x_0, of 0.2. Then enter: $y_1(Ans)$. You repeat $y_1(Ans)$, and this will input the previous number obtained into the cosine function. You repeat until you converge to a number. Again, for certain functions there will not be a fixed point.

$$x_0 = 0.2$$
$$x_1 = \cos(0.2) = 0.9800665778$$
$$x_2 = \cos(0.9800665778) = 0.5569672528$$
$$x_3 = \cos(0.5569672528) = 0.8488621657$$
$$x_4 = \cos(0.8488621657) = 0.6608375511$$
$$x_5 = \cos(0.6608375511) = 0.7894784378$$
$$x_6 = cos(0.7894784378) = 0.7042157133$$
$$x_7 = cos(0.7042157133) = 0.7621195618$$
$$x_8 = cos(0.7621195618) = 0.7233741721$$
$$x_9 = \cos(0.7233741721) = 0.7495765763$$
$$x_{10} = \cos(0.7495765763) = 0.7319774253$$
$$x_{11} = cos(0.7319774253) \quad = 0.7438542615$$
$$x_{12} = cos(0.7438542615) = 0.7358641982$$
$$x_{13} = cos(0.7358641982) = 0.7412509563$$
$$x_{14} = cos(0.7412509563) = 0.7376244765$$
$$x_{15} = cos(0.7376244765) = 0.7400682604$$
$$x_{16} = cos(0.7400682604) = 0.7384225298$$
$$x_{17} = cos(0.7384225298) = 0.7395313085$$

$$x_{18} = cos(0.7395313085) = 0.7387845106$$

$$x_{19}\, cos(0.7387845106) = 0.7392876028$$

$$x_{20} = cos(0.7392876028) = 0.7390851332$$

$$\dots$$

$$x_{58} = 0.7390851332$$

$$x_{59} = 0.7390851332$$

So the sequence $\{\, x_1,\ x_2,\ x_{3,}\, x_4, \dots \}$ converges to 0.7390851332 which is an approximation to the equation $x = cosx$ correct to ten decimal places.

I would explain that the method will not always work. Some equations will diverge and will not converge to a root. I show that the example of $x = 1.8cosx$, starting with an initial estimate of a solution of $x = 1.0$. After six iterations below, it is clear that the numbers are not getting close to approximating a root:

$$1.8cos(1.0) = 0.9725441506$$

$$cos(0.9725441506) = 1.013758327$$

$$cos(1.013758327) = 0.9516137432$$

$$cos(0.9516137432) = 1.04466544$$

$$cos(1.04466544) = 0.9039442812$$

$$cos(0.9039442812) = 1.113327863$$

$$cos(1.113327863) = 0.7950209356$$

The following is a lesson problem for students to work through on their own or in a small group:

Problem: Find the all of the roots of the polynomial $x^3 - 3x + 1 = 0$ using the fixed point method. Verify answers by graphing and finding the zeros.

Solution: First the equation should be solved: $x = \sqrt[3]{3x - 1}$. After 37 iterations, the first solution using an initial guess of $x_0 = 1.0$ is $x = 1.532088886$.

The second solution using an initial guess of $x_0 = 0$ is $x = -1.879385242$. This is obtained after 24 iterations. Since the equation is a cubic, a third solution must be obtained. The equation should be solved: $x = \frac{-1}{x^2 - 3}$. The third solution using an initial guess of $x_0 = 1.0$ is $x = 0.3472963553$. This is obtained after 13 iterations.

VIII. Newton's Method of Approximating Roots

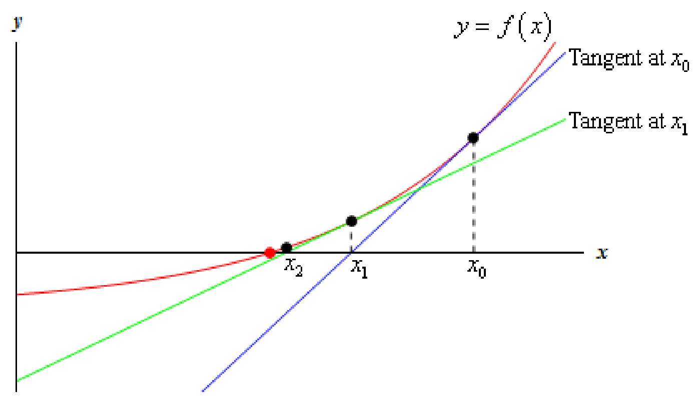
a. Mathematical Background of Newton's Method

Newton's Method, also known as the Newton-Raphson Method, is a numerical analysis method for approximating roots for an equation. It is similar to the fixed-point method as it employs a root-finding algorithm that successively finds better approximations to the roots of an equation. The method is named after Isaac Newton and Joseph Raphson. According to Struik, the method was originally found in Newton's "*De analysis per aequationes numero terminorum infinitas*" and "*De metodis fluxionum et serierum infinitarum*", written in 1669 and 1671 respectively. However, the method as it is utilized today differs from Newton's original method where he computes a sequence of polynomials and at the end arrives at an approximation for the root(s). In addition, Newton viewed the method as purely algebraic and did not connect the method to calculus. Newton's Method was first published in 1685 by John Wallis in a "*Treatise of Algebra both Historical and Practical*". Then in 1690, Joseph Raphson published a simplified description of the method in "*Analysis aequationum universalis*". Raphson described the method in a manner that is more similar to the method as it is typically employed today by using successive approximations of the root(s).

b. Lesson on Newton's Method

Newton's Method can be taught to students in a first-year calculus course. Students will need to familiar with the definition of a derivative and methods of calculating derivatives. I would explain the history behind the method and focus on the iterative nature of the method. As illustrated in Diagram 6, I would draw the graph of $f(x)$ and choose an approximation x_0 to the equation $f(x) = 0$. Then I would draw the tangent line to the curve $y = f(x)$ at $(x_0, f(x_0))$ whose equation is $y = f'(x_0)(x - x_0) + f(x_0)$. I would call the *x*-intercept of this tangent line x_1, where $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$. Next I would draw the tangent line to the curve $y = f(x)$ at $(x_1, f(x_1))$ and find its *x*-intercept x_2, where $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$. Then we start the iterative process and have $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

Diagram 5



I would work through the lesson example to find the roots of: $f(x) = cosx - x$. The graphing calculator would be utilized as follows:

1. $y_1 = cosx - x$
2. Compute the derivative, $y_1'(x)$, and input: $y_2 = -sinx - 1$
3. $y_3 = x - \frac{y_1}{y_2}$
4. We would choose 1, a value in between 0 and $\frac{\pi}{2}$, as the initial value.
5. After 4 iterations, inputted as $y_3(Ans)$, the answer is repeated and the root can be approximated at 0.7390851332. This answer can be verified by finding the zero on the graphing calculator.

Lesson problem for students to work through on their own or in a small group:

Find the largest root of the polynomial equation $x^3 - 3x + 1 = 0$ using Newton's method. Verify the answer by graphing.

Solution:

1. $y_1 = x^3 - 3x + 1$
2. Compute the derivative, $y_1'(x)$, and input: $y_2 = 3x^2 - 3$
3. $y_3 = x - \frac{y_1}{y_2}$
4. Choose 2 as the initial value.
5. After 6 iterations, inputted as $y_3(Ans)$, the answer is repeated and the root can be approximated at 1.532088886.

IX. Gauss's Easter Algorithm

a. Mathematical Background of Gauss's Easter Algorithm

As strange as it may seem, history reports that Gauss grew up never knowing the exact date of his birth. His mother always told him that his birthday was shortly after Easter in 1777. Specifically, she knew that Gauss was born on the Wednesday before the Ascension, which is the day Christians believe that Jesus Christ ascended into heaven. Ascension Day is always forty days following Easter. For the Western (Gregorian) calendar, Easter is determined based upon the solar calendar and the phases of the moon. There is approximately a five-week window during which Easter can fall in any particular calendar year. The earliest recorded Easter date in recent history was 3/22/18 and the latest was on 4/25/43. Using the Gregorian calendar, Easter is on the first Sunday after the first full moon following the spring equinox. Gauss followed the Gregorian calendar. The Orthodox (Julian) calendar typically places Easter about two weeks after the Gregorian calendar each year, but the calendar dates for Easter can match during certain years or differ by as much as five weeks. Gauss, being a most astute mathematician, set out to determine what day Easter occurred in the year 1777. If he could figure out that date, he would know the day of his birth.

Gauss was confident that with some initial information, he would be able to create an algorithm that would determine the exact date of Easter of any particular calendar year. As stated by Trent, Gauss was able to create an algorithm that would calculate the dates of Easter between the years 1582 and 2099. Gauss researched numerous old calendars and he developed Table 3 to assist in his calculations.

Table 3

Years	**M**	**N**
1582-1699	22	2
1700-1799	23	3
1800-1899	23	4
1900-2099	24	5

Furthermore, Gauss's methodology was based upon remainders after dividing by seven, for the number of days in a week and dividing by 30, since there is an average of 30 days in a month, and dividing by four, for the number of weeks in a month. The remainders are how he accounts for the phases of the sun and the moon. This "remainder logic" was actually part of a new branch of number theory that Gauss called "modular arithmetic". The modern approach to modular arithmetic was developed by Gauss in his book '*Disquisitiones Arithmeticae'*, which was published in 1801.

Modular arithmetic is also referred to as "Clock Arithmetic", as a 12-hour clock uses this methodology in which a day is divided into two 12-hour periods. For example, you can calculate a time of 3:00 pm by taking 15, the number of hours that have passed since 12:00 am and dividing by 12. Fifteen divided by 12 provides a remainder of three, so that determines a time of

3:00 pm. In another example utilizing modular arithmetic, 50 would be congruent to 14 with a modulo, or *mod*, of 12. This is true because $50 - 14 = 36$, and 36 is a multiple of 12. This is written as $50 \equiv 14\ (mod\ 12)$. This is also the case using modular arithmetic for negative numbers. For example, $-13 \equiv 7\ (mod\ 5)$, as $-13 - 7 = -20$, and -20 is a multiple of 5. Modular arithmetic was also used in Section V to determine i raised up to any whole number exponent.

Utilizing modular arithmetic combined with his research, Gauss generated Table 4 of formulas that determined his birthday was Wednesday 4/30/1777! I will step the students through his exact calculations using the tables of information he created to determine Gauss's date of birth in the next lesson.

Table 4

Year divided by four	Remainder = a
Year divided by seven	Remainder = b
Year divided by 19	Remainder = c
$\frac{19c + M}{30}$	Remainder = d
$\frac{2a + 4b + 6d + N}{7}$	Remainder = e
$22 + d + e =$	Date in March
$d + e - 9 =$	Date in April

b. Lesson on Gauss's Easter Algorithm

After explaining the background on Gauss's birthday dilemma, we will use the formulas in Tables 3 and 4 to determine the date of Easter in 1777, and then Gauss's birthday. This lesson is very versatile, as it could be used with students with varying levels of mathematical ability. Students in an Algebra 1 class would be ideal for utilizing this algorithm, however students with more or less sophisticated mathematical skills could enjoy this lesson. It could also be used as a review of long division, potentially in preparation for long division of polynomials.

Steps:

1. Using Table 3, the year 1777 results in $M = 23, N = 3$
2. The next five calculations are based on the remainders generated using Table 4.
   a. Divide 1777 by 4, which is the number of weeks in a month. $\frac{1777}{4} = 444,$ $remainder$ 1. So $a = 1$.
   b. Divide 1777 by 7, which is the number of days in a week. $\frac{1777}{7} = 253,$ $remainder$ 6. So $b = 6$.
   c. Divide 1777 by 19, which represents a 19 year cycle in the lunar calendar. $\frac{1777}{19} = 93,\ remainder$ 10. So $c = 10$.
   d. Using the formula found in the fourth row of Table 4 and substituting 10 in for c and 23 in for M, $\frac{19c + M}{30} = \frac{19(10) + 23}{30} = \frac{213}{30} = 7, remainder$ 3. So $d = 3$.

e. Using the formula found in the fifth row of Table 4 and substituting 1 in for a, 6 in for b, 3 in for d, and 3 in for N : $\frac{2a+4b+6d\,+N}{7} = \frac{2(1)\,+\,4(6)\,+6(3)\,+\,3}{7} = \frac{47}{7} = 6, remainder\ 5$. So $e = 5$.

f. The Easter date is found by using one of the two formulas in the last two rows of the Table 4. We will not use the April formula, as using the April formula would result in a negative day: $d + e - 9 = 3 + 5 - 9 = -1$. So, we use the March formula: $22 + d + e = 30$. Based on the algorithm, Easter in year 1777 was on March 30th.

g. Finally, since Gauss's birthday was the Wednesday during the week before the Ascension, which is forty days following Easter, we count 39 days after Easter (as Easter is included in the forty days). Counting one more day in March, 30 days in April and then eight days in May, Ascension Day was May 8th, 1777. Then the Wednesday in the week before May 8th, 1777 was April 30th 1777. So Gauss was born on April 30th 1777.

After working through the problem, I would then have the students figure out the date that Easter will be on in the year 2050. Students will work together in small groups or individually to solve the problem.

Steps:

1. Using Table 3, the year 2050 results in $M = 24, N = 5$

2. Steps 2-7 use Table 4. Divide 2050 by 4, $\frac{2050}{4} = 512,\ remainder\ 2$. So $a = 2$.

3. Divide 2050 by 7, $\frac{2050}{7} = 292,\ remainder\ 6$. So $b = 6$.

4. Divide 2050 by 19, $\frac{2050}{19} = 107,\ remainder\ 17$. So $c = 17$.

5. $\frac{19c\,+\,M}{30} = \frac{19(17)\,+\,24}{30} = \frac{347}{30} = 11,\ remainder\ 17$. So $d = 17$.

6. $\frac{2a+4b+6d\,+\,N}{7} = \frac{2(2)\,+\,4(6)\,+6(17)\,+\,5}{7} = \frac{135}{7} = 19,\ remainder\ 2$. So $e = 2$.

7. Using the April formula: $d + e - 9 = 17 + 2 - 9 = 10$. So, according to Gauss's algorithm, Easter in 2050 will be on April 10^{th}. This date can be validated by consulting tables on the Internet.

If time allows, I would have the students compute the dates for Easter in the following years, past and present: 1945: April 1^{st}, 1910: March 27^{th} and 2095: April 24^{th}. The April 24^{th} 2095 date happens to be the second latest recorded date for Easter. This date has only previously occurred in 1859 and 2011. Students could also pick their own years that they may have a particular interest in, compute the dates using the algorithm, and then independently verify their answers.

X. Conclusion

Algebra is an incredibly diverse and complex field of mathematics. Students can often think that if they are easily able to solve a two or three step linear equation and graph a line, they are experts in algebra. Algebra involves so much more than a simple linear equation! Algebra has an intricate past and can be traced back to a 9^{th} century manuscript written by the Persian scholar Al-Khwarizmi, who is often referred to as the 'father of Algebra', and the word itself stems from the Arabic word *'al-jabr'*. Introducing students to Gauss and other mathematical giants such as Al-Khwarizmi, Descartes and Newton can enlighten students to the multifaceted history of this exciting mathematical field.

Knowing the historic roots of a topic that is being studied provides students with an important foundation for the various problem-solving techniques and methods that they will be utilizing in their mathematics course. When students realize that civilization has been challenged with solving the same types of problems that they are working on for hundreds or even thousands of years, it can ignite a certain interest or passion within. Students must be continually motivated in new and innovative ways, especially when the mathematics students are expected to learn becomes particularly tough or challenging beyond a level that they have encountered before. By introducing historical aspects of mathematics and brilliant mathematical minds of the past, I hope to instill a sense of pride and empowerment in my students as they become astute problem-solvers that are equipped to confront the intense challenges of the future world we will live in.

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