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USING GEOGEBRA TO EXPLORE PROPERTIES OF CIRCLES IN EUCLIDEAN GEOMETRY

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Erin E. Hanna
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The essay of Erin E. Hanna is hereby accepted:

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Advisor – Barbara K. D'Ambrosia Date

I certify that this is the original document

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Author – Erin E. Hanna Date

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Introduction

In teaching geometry for the past four years, I have noticed particular units of material that students traditionally struggle with. One of these units is the unit on circles because it is bogged down with a lot of definitions and theorems for students to remember. In the past, I have given students various definitions and theorems to look over each day in a rather boring, mundane way. After considering the definitions and theorems, we would discuss them and apply them to problems, but never really explore them or figure out how they work.

When I discovered the program GeoGebra, my ideas for teaching the unit on circles changed. I recognized GeoGebra to be a dynamic program that would allow my students to design diagrams that fit the criteria of each definition or theorem, and would require them to manipulate and analyze their diagrams to prove each theorem rather than simply memorize information and parrot it back to me. As a result, the lessons in this essay were designed to allow students to explore the unit on circles in a more engaging way. In addition, I chose GeoGebra because my students love to work with technology. It is my hope that bringing these geometric ideas right to their laptops will help students stay engaged and enjoy what they are learning. The lessons in this paper can be modified for use with other dynamic geometry platforms, but GeoGebra has the advantage of being free and since it is a mainstream platform, students may encounter it in later courses.

The lessons included in the unit on circles include basic definitions of parts of circles and theorems about: tangents; arcs and their central angles; arcs and their chords; inscribed angles; other angles formed by the intersection of two secants, two tangents, or a secant and a tangent; the lengths of different segments in circles.

I believe it is ideal to allow students to work in pairs for each lesson so they can help each other through the material and ensure that things are done correctly. Working with a partner is especially helpful when creating an accurate diagram using GeoGebra based on the steps provided in each lesson. Students will learn quickly that the key to a successful discovery is the creation of a diagram that matches all of the necessary criteria. Although I have not used this set of lessons yet with students, I plan to implement them this year in my teaching and hope for strong results.

It is important to note that various versions of GeoGebra exist and each one is slightly different from others. For my lessons, I chose to use the classic web version of GeoGebra that is available at geogebra.org/classic. Since my students have a variety of technology devices, I found the classic web version of GeoGebra to be the most standard and easily accessible option for students. Also, because it is web-based, students can access GeoGebra at home.

For each lesson in this paper, I have included detailed teacher notes with required pre-requisite knowledge of other geometric concepts along with proofs of theorems, an annotated student document, and a blank student document. There is also an appendix at the end of the paper that gives more information on where to find each GeoGebra tool on the classic web version of GeoGebra and a short description of how to use each tool. It is assumed that students have some knowledge of GeoGebra before completing the lessons in this paper, including how to turn off the grid and axes, how to relabel objects, and how to hide objects.

Lesson 1: Basic Terms
Teacher Overview

Introduction: This lesson helps students to identify the basic parts of a circle that will be referenced throughout the remaining lessons, and to explore various tools in GeoGebra.

Pre-requisite Knowledge: none

Allotted Time: 1 class period (40 minutes)

Materials:

- For Students
    - Student Handout (1 per student)
    - Technology device with internet capability (at least 1 per pair of students)
- For Teacher
    - Teacher Notes
    - Annotated Student Handout

Lesson Overview:

- Students work in pairs on Student Handout. (30 minutes)
    - Monitor student work by circulating around the classroom. If students are struggling, ask pointed questions to get them back on track. Refrain from giving students direct answers, as this is meant to be a discovery activity.
    - Once their figures have been checked, students can work on the sample problems to practice the concepts they have learned.
- Bring students back into a whole group setting and go over the sample problems. Then assign homework problems related to the lesson content. (10 minutes)

Lesson 1: Basic Terms
Annotated Student Document

The focus in this unit is the geometric figure called a circle. Today, we will introduce many terms that relate to a circle and visualize these terms using GeoGebra.

Part 1

- ☐ Go to www.geogebra.org/classic. Be sure to turn off the grid and axes features to declutter your screen.

- ☐ Using the Point tool, select a location for point *A*.

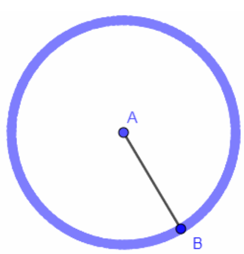
- ☐ Suppose you want to find a point *B* that is 3 units away from *A*. Using the Segment with Given Length tool, we can create a segment that connects *A* to *B* and that has a length of 3 units. GeoGebra placed *B* in a particular location. Are there other locations for *B* that are 3 units away from *A*? How many different points *B* are 3 units from *A*?

  Students may take the time to conjecture here. Most will agree that there are infinitely many points *B* that are 3 units from *A*.

- ☐ Right click on *B* and select the "Show Trace" button. Using the Move tool, move *B* around. As you do, note that the distance from *A* to *B* remains 3 units. What do you notice about the trace of *B*?

  Students should note that the trace of *B* starts to form a circle.

  At this point, your figure should look similar to this, although your points may be elsewhere on the graph.



Moving *B* around *A* while keeping the distance from *A* the same results in a **circle**. For this reason, we define a circle to be the set of points in a plane at a given distance from a fixed point in that plane.

The fixed point, which we've called *A*, is the **center** of the circle and the given distance, which is AB in our case, is the **radius** of the circle. We also use the word **radius** to refer to any segment that joins the center of a circle to a point on the circle. The plural of radius is **radii**.

What, therefore, can you assume is true about all radii of a circle?

All radii of a circle are congruent.

We sometimes refer to a circle by its center. In the example above, we are working with $\odot A$.

Part 2

☐ Open a new GeoGebra window.

☐ Using the [Circle with Center through Point] tool, create circle *c* with center *A* and point *H* on the circle. Then hide *H*.

☐ Using the [Point on Object] tool, put points *B*, *C*, and *D* on *c*. Then using the Segment tool, connect *A* and *B* to form $\overline{AB}$, and connect *C* and *D* to form $\overline{CD}$.

Which of these segments is a radius? $\overline{AB}$

We call $\overline{CD}$ a **chord**, which is a segment whose endpoints lie on a circle.

☐ Using the [Line] tool, create line *h* through *C* and *D*.

Note that $\overleftrightarrow{CD}$ is not a chord. Why? It is not a segment. It has no endpoints.

We call $\overleftrightarrow{CD}$ a **secant**, which is a line that contains a chord.

☐ Using the [Point on Object] tool, put point *E* on *c*. Then using the [Line] tool, create line *i* through *E* and *A*.

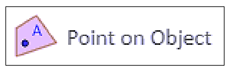
☐ Using the [Intersect] tool, find the other intersection of $\overleftrightarrow{AE}$ and *c*. Label this point *F*.

Note that $\overline{EF}$ is a chord. Why? It is a segment whose endpoints lie on a circle.

In fact, $\overline{EF}$ is a special chord because it goes through *A*, the center of the circle. This type of chord is called a **diameter**.

How does a diameter relate to a radius?

A diameter is made up of two different radii that lie on the same line. The diameter's length is twice the length of a radius.

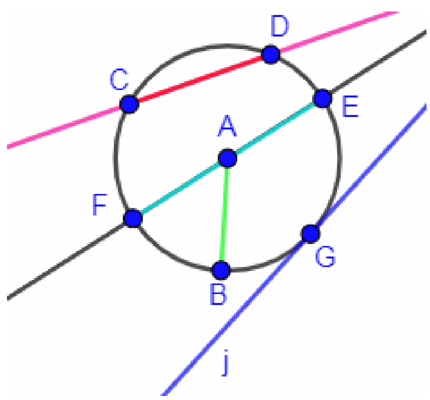
☐ Using the 

tool, put point *G* on *c*. Then using the 

tool, create a line *j* through *G*.

This line *j* is a special type of line called a **tangent**. It intersects the circle in exactly one point, *G*, which is referred to as the **point of tangency**.

At this point, your figure should look similar to the diagram below, although your points and lines may be elsewhere on the circle. Be sure your circle includes radius $\overline{AB}$, chord $\overline{CD}$, secant $\overleftrightarrow{CD}$, diameter $\overline{EF}$, and tangent *j* with point of tangency *G*.



Have your teacher check your figure before moving on!

Use the basic terms defined in this section to determine if each statement is *true* or *false*. Explain your reasoning.

1. If $\overline{OA}$ is a radius of $\odot O$ and $\overline{NB}$ is a radius of $\odot N$, then $\overline{OA} \cong \overline{NB}$.

   False; they are radii of two different circles so they don't have to be congruent.

2. A radius of a circle is also a chord.

   False; a radius only has one endpoint on the circle, not two.

3. A diameter is always twice the length of a radius.

   True; a diameter is comprised of two radii.

4. A tangent intersects a circle in more than one point.

   False; a tangent intersects a circle in exactly one point. If a line intersects a circle in more than one point, then it is a secant rather than a tangent.

5. A point of tangency must be on the circle.

   True; the point of tangency is the one point at which a tangent line intersects a circle.

6. You can find the length of a secant.

   False; a secant is a line, so it extends infinitely. You can, however, find the length of the chord that the secant contains.

7. The longest chords in a circle are the diameters.

   True; any chord that is not a diameter does not go through the center of the circle, so we can use two radii to create a triangle with the chord as the third side. Thus, the length of the chord is less than the sum of the lengths of the two radii. Since a diameter is comprised of two radii, it must be the longest chord.

Lesson 1: Basic Terms
Student Document

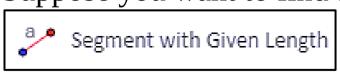
The focus in this unit is the geometric figure called a circle. Today, we will introduce many terms that relate to a circle and visualize these terms using GeoGebra.

Part 1

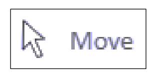
- ☐ Go to www.geogebra.org/classic. Be sure to turn off the grid and axes features to declutter your screen.

- ☐ Using the 

tool, select a location for point *A*.

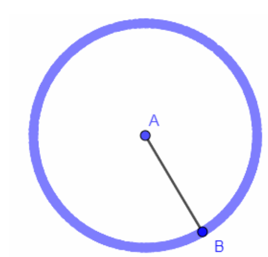
- ☐ Suppose you want to find a point *B* that is 3 units away from *A*. Using the 

tool, we can create a segment that connects *A* to *B* and that has a length of 3 units. GeoGebra placed *B* in a particular location. Are there other locations for *B* that are 3 units away from *A*? How many different points *B* are 3 units from *A*?

- ☐ Right click on *B* and select the "Show Trace" button. Using the 

tool, move *B* around. As you do, note that the distance from *A* to *B* remains 3 units. What do you notice about the trace of *B*?

At this point, your figure should look similar to this, although your points may be elsewhere on the graph.



Moving *B* around *A* while keeping the distance from *A* the same results in a **circle**. For this reason, we define a circle to be the set of points in a plane at a given distance from a fixed point in that plane.

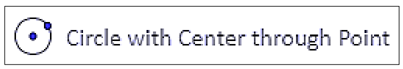
The fixed point, which we've called *A*, is the **center** of the circle and the given distance, which is AB in our case, is the **radius** of the circle. We also use the word **radius** to refer to any segment that joins the center of a circle to a point on the circle. The plural of radius is **radii**.

What, therefore, can you assume is true about all radii of a circle?

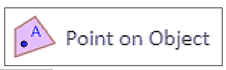
We sometimes refer to a circle by its center. In the example above, we are working with $\odot A$.

Part 2

☐ Open a new GeoGebra window.

☐ Using the 

tool, create circle *c* with center *A* and point *H* on the circle. Then hide *H*.

☐ Using the 

tool, put points *B*, *C*, and *D* on *c*. Then using the 

tool, connect *A* and *B* to form $\overline{AB}$, and connect *C* and *D* to form $\overline{CD}$.

Which of these segments is a radius?

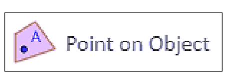
We call $\overline{CD}$ a **chord**, which is a segment whose endpoints lie on a circle.

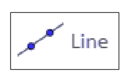
☐ Using the 

tool, create line *h* through *C* and *D*.

Note that $\overleftrightarrow{CD}$ is not a chord. Why?

We call $\overleftrightarrow{CD}$ a **secant**, which is a line that contains a chord.

☐ Using the 

tool, put point *E* on *c*. Then using the 

tool, create line *i* through *E* and *A*.

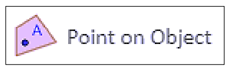
☐ Using the 

tool, find the other intersection of $\overleftrightarrow{AE}$ and *c*. Label this point *F*.

Note that $\overline{EF}$ is a chord. Why?

In fact, $\overline{EF}$ is a special chord because it goes through *A*, the center of the circle. This type of chord is called a **diameter**.

How does a diameter relate to a radius?

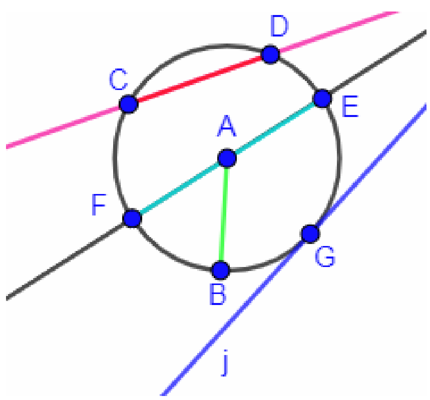
☐ Using the 

tool, put point *G* on *c*. Then using the 

tool, create a line *j* through *G*.

This line *j* is a special type of line called a **tangent**. It intersects the circle in exactly one point, *G*, which is referred to as the **point of tangency**.

At this point, your figure should look similar to the diagram below, although your points and lines may be elsewhere on the circle. Be sure your circle includes radius $\overline{AB}$, chord $\overline{CD}$, secant $\overleftrightarrow{CD}$, diameter $\overline{EF}$, and tangent *j* with point of tangency *G*.



STOP Have your teacher check your figure before moving on!

Use the basic terms defined in this section to determine if each statement is *true* or *false*. Explain your reasoning.

1. If $\overline{OA}$ is a radius of $\odot O$ and $\overline{NB}$ is a radius of $\odot N$, then $\overline{OA} \cong \overline{NB}$.

2. A radius of a circle is also a chord.

3. A diameter is always twice the length of a radius.

4. A tangent intersects a circle in more than one point.

5. A point of tangency must be on the circle.

6. You can find the length of a secant.

7. The longest chords in a circle are the diameters.

Lesson 2: Tangents
Teacher Overview

Introduction: This lesson helps students to become more familiar with the concept of a tangent to a circle, and to explore the following important theorem and corollary related to tangents. Students will work in pairs, using GeoGebra.

- Theorem: A tangent line to a circle is perpendicular to the radius drawn to the point of tangency.

- Corollary: Tangent segments from a common point external to a circle are congruent.

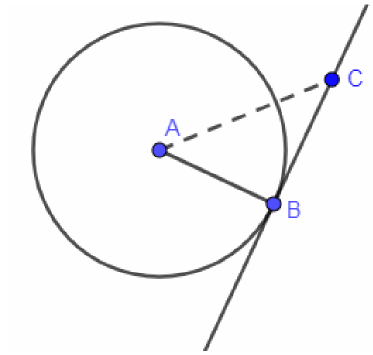
Pre-requisite Knowledge:
- Shortest distance from a point to a line is the perpendicular distance
- Hypotenuse-Leg triangle congruence

Allotted Time: 1 class period (40 minutes)

Materials:
- For Students
  - Student Handout (1 per student)
  - Technology device with internet capability (at least 1 per pair of students)
- For Teacher
  - Teacher Notes
  - Annotated Student Handout

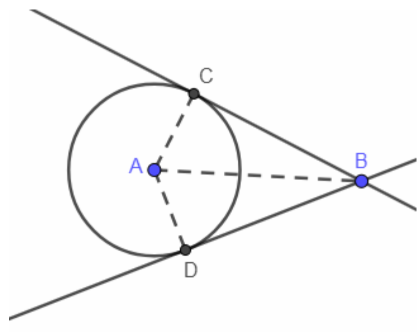
Lesson Overview:
- Students work in pairs on Student Handout. (25 minutes)
  - Monitor student work by circulating around the classroom. If students are struggling, ask pointed questions to get them back on track. Refrain from giving students direct answers, as this is meant to be a discovery activity.
  - As students form their conjectures, check to make sure the conjectures are accurate. If not, redirect students in a helpful manner to allow them to find the correct conclusions.
  - Once their conjectures have been checked, students can work on the sample problems to practice the concepts they learned.
- Bring students back into a whole group setting and go over the proofs of these conjectures. (10 minutes)
  - Use the proofs on the following page to lead the discussion, but ask for student input along the way.
  - Students should follow along and take notes on the proofs if necessary.
- Conclude the lesson by reviewing the sample problems and assigning homework problems related to the lesson content. (5 minutes)

Theorem: If a line is tangent to a circle, then the line is perpendicular to the radius drawn to the point of tangency.



Proof: Consider a line that is tangent to $\odot A$ at *B*. Assume temporarily that this line is not perpendicular to radius $\overline{AB}$. Then the perpendicular segment from *A* to the tangent line intersects the tangent in some other point, say *C*. Draw $\overline{AC}$. The shortest segment from a point to a line is the perpendicular segment, so $AC < AB$. Note that tangent $\overleftrightarrow{BC}$ intersects the circle only at point *B*. So *C* lies outside the circle and thus AC is greater than the radius AB. This is a contradiction to the statement that $AC < AB$, so our temporary assumption must be false. Therefore, $\overleftrightarrow{BC} \perp \overline{AB}$.

Corollary: Tangent segments from a common point external to a circle are congruent.



Proof: Let *B* be a point outside a circle with center *A*. Let $\overleftrightarrow{BC}$ and $\overleftrightarrow{BD}$ be tangents to the circle at points *C* and *D*. Draw radii $\overline{AC}$ and $\overline{AD}$, and segment $\overline{AB}$. Then by the theorem above, $\overline{BC} \perp \overline{AC}$ and $\overline{BD} \perp \overline{AD}$. This implies that $m\angle BCA = m\angle BDA = 90°$, and ΔBCA and ΔBDA are right triangles. Since $\overline{AC}$ and $\overline{AD}$ are radii of the circle, $\overline{AC} \cong \overline{AD}$. Also, $\overline{AB} \cong \overline{AB}$, so $\Delta BCA \cong \Delta BDA$ by Hypotenuse-Leg Congruence. Thus $\overline{BC} \cong \overline{BD}$.

Lesson 2: Tangents
Annotated Student Document

What is the definition of a tangent to a circle?

A tangent to a circle is a line that intersects the circle in **exactly** one point.

Now we will explore tangents more and discover some of their useful properties.

Part 1

- ☐ Go to www.geogebra.org/classic. Be sure to turn off the grid and axes features to declutter your screen.

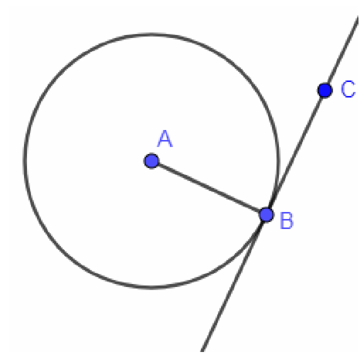
- ☐ Using the [Circle with Center through Point] tool, create circle *c* with center *A* and point *D* on the circle. Then hide *D*.

- ☐ Using the [Point on Object] tool, put a point *B* on *c*.

- ☐ Using the [Tangents] tool, create a line *f* that goes through *B* and is tangent to *c*. Then using the [Point on Object] tool, put a point *C* on *f*.

- ☐ Using the Segment tool, connect *A* and *B* to form $\overline{AB}$.

  At this point, your figure should look similar to this, although your tangent may be at a different point on the circle.



What is the segment $\overline{AB}$ called? A radius of *c*

☐ We will explore the measure of $\angle ABC$, the angle formed by $\overline{AB}$ and *f*.

Using the Angle tool, measure $\angle ABC$ and record its measure below.

$m\angle ABC = 90°$

☐ Then use the Move tool to move point *B* around circle *c*.

What do you notice about the measure of $\angle ABC$ as you move point *B*? What does this imply about *f* (a tangent line) and $\overline{AB}$ (a radius drawn to the point of tangency)? Fill in the conjecture statement below appropriately:

Students should notice that moving *B* does not affect the measure of $\angle ABC$. No matter where *B* falls on the circle, $m\angle ABC = 90°$.

If a line is tangent to a circle, then the line is perpendicular to the radius drawn to the point of tangency.

☐ Unhide *D*, the point that you used to create the circle, and move it to change the size of the circle. Does the conjecture still hold?

Students should note that the conjecture still holds. The radius is still perpendicular to the tangent line at its point of tangency.

STOP Have your teacher check your conjecture before moving on!

Part 2

☐ Open a new GeoGebra window.

☐ Using the Circle with Center through Point tool, create circle *c* with center *A* and point *E* on the circle. Then hide *E*.

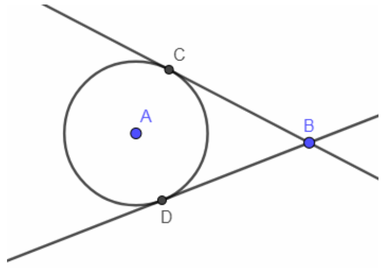
☐ Using the Point tool, create point *B* anywhere outside of *c*.

☐ Using the Tangents tool, create two lines *f* and *g* that are tangent to *c* and go through *B*.

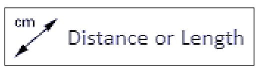
☐ Using the 

tool, create points of intersection *C* and *D* where lines *f* and *g* meet *c*.

Your figure should look similar to this, although *B* may be closer or farther from the circle.

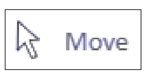


We define $\overline{BC}$ and $\overline{BD}$ to be **tangent segments**, which are segments of tangent lines that have one endpoint at the point of tangency and the other endpoint on the tangent line. We will now explore the lengths of tangent segments $\overline{BC}$ and $\overline{BD}$.

☐ Using the 

tool, measure the lengths of $\overline{BC}$ and $\overline{BD}$. What do you notice?

Students should notice that the segments have equal lengths, so $\overline{BC} \cong \overline{BD}$.

☐ Then use the 

tool to move *B* closer to and farther away from *c*.

Does your observation above still hold?

Students should note that the observation still holds, no matter how far *B* is from *c*.

☐ Finally, unhide *E*, the point that you used to create the circle. Move *E* to change the size of the circle. Does your observation above still hold? What does this imply about two tangent segments to a circle from the same point? Fill in the conjecture statement appropriately:

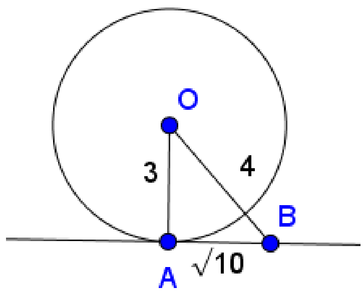
Students should note that the observation still holds, no matter the size of the radius of *c* or the distance between *B* and *c*.

Tangent segments from a common point external to a circle are congruent.

STOP Have your teacher check your conjecture before moving on!

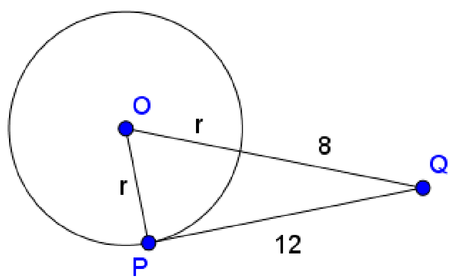
Use your conjectures to answer the following questions.

1. Is $\overleftrightarrow{AB}$ tangent to $\odot O$?



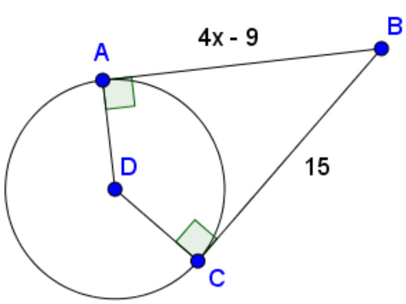
If $\overleftrightarrow{AB}$ is a tangent line, then $m\angle OAB = 90°$ and ΔOAB would have be a right triangle. Notice that $3^2 + \left(\sqrt{10}\right)^2 = 9 + 10 = 19$ but $4^2 = 16$. Since $19 \neq 16$, ΔOAB is not a right triangle. Thus, $\overleftrightarrow{AB}$ is not tangent to $\odot O$.

2. If $\overline{QP}$ is tangent to $\odot O$ at *P*, what is the radius of $\odot O$?



Since $\overline{QP}$ is tangent to $\odot O$ at *P*, $m\angle QPO = 90°$ and ΔQPO is a right triangle. By the Pythagorean Theorem, $r^2 + 12^2 = (r+8)^2$, so $r = 5$.

3. If $\overline{AB}$ and $\overline{CB}$ are tangents to $\odot D$, find the value of *x*.



$\overline{AB}$ and $\overline{CB}$ are tangents to a circle from the same point, so they must be congruent. Thus, $4x - 9 = 15$ and $x = 6$.

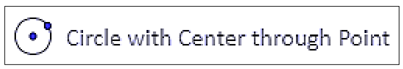
Lesson 2: Tangents
Student Document

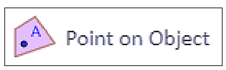
What is the definition of a tangent to a circle?

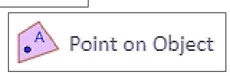
Now we will explore tangents more and discover some of their useful properties.

Part 1

- ☐ Go to www.geogebra.org/classic. Be sure to turn off the grid and axes features to declutter your screen.

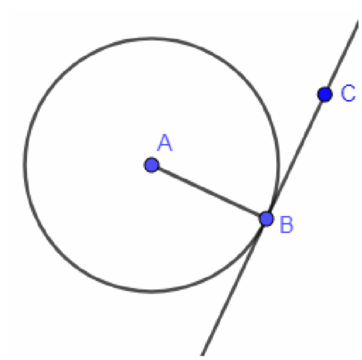
- ☐ Using the  tool, create circle *c* with center *A* and point *D* on the circle. Then hide *D*.

- ☐ Using the  tool, put a point *B* on *c*.

- ☐ Using the  tool, create a line *f* that goes through *B* and is tangent to *c*. Then using the  tool, put a point *C* on *f*.

- ☐ Using the  tool, connect *A* and *B* to form $\overline{AB}$.

  At this point, your figure should look similar to this, although your tangent may be at a different point on the circle.



What is the segment $\overline{AB}$ called?

- ☐ We will explore the measure of $\angle ABC$, the angle formed by $\overline{AB}$ and f.

  Using the Angle tool, measure $\angle ABC$ and record its measure below.

  $m\angle ABC =$ ______

- ☐ Then use the Move tool to move point B around circle c.

  What do you notice about the measure of $\angle ABC$ as you move point B? What does this imply about f (a tangent line) and $\overline{AB}$ (a radius drawn to the point of tangency)? Fill in the conjecture statement below appropriately:

  If a line is tangent to a circle, then the line is ____________________________ the radius drawn to the point of tangency.

- ☐ Unhide D, the point that you used to create the circle, and move it to change the size of the circle. Does the conjecture still hold?

STOP Have your teacher check your conjecture before moving on!

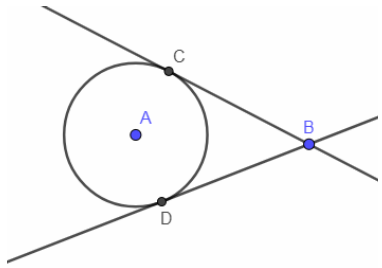
Part 2

- ☐ Open a new GeoGebra window.
- ☐ Using the Circle with Center through Point tool, create circle c with center A and point E on the circle. Then hide E.
- ☐ Using the Point tool, create point B anywhere outside of c.
- ☐ Using the Tangents tool, create two lines f and g that are tangent to c and go through B.

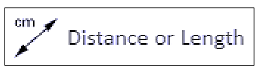
☐ Using the 

tool, create points of intersection C and D where lines f and g meet c.

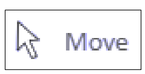
Your figure should look similar to this, although B may be closer or farther from the circle.



We define $\overline{BC}$ and $\overline{BD}$ to be **tangent segments**, which are segments of tangent lines that have one endpoint at the point of tangency and the other endpoint on the tangent line. We will now explore the lengths of tangent segments $\overline{BC}$ and $\overline{BD}$.

☐ Using the 

tool, measure the lengths of $\overline{BC}$ and $\overline{BD}$. What do you notice?

☐ Then use the 

tool to move B closer to and farther away from c.

Does your observation above still hold?

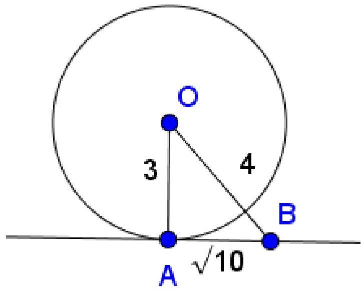
☐ Finally, unhide E, the point that you used to create the circle. Move E to change the size of the circle. Does your observation above still hold? What does this imply about two tangent segments to a circle from the same point? Fill in the conjecture statement appropriately:

Tangent segments from a common point external to a circle are __________.

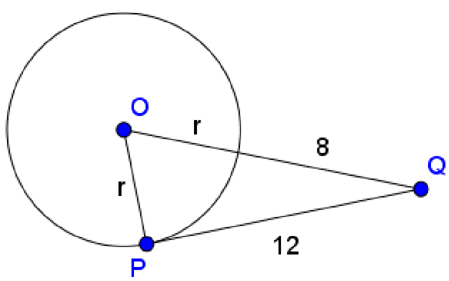
Have your teacher check your conjecture before moving on!

Use your conjectures to answer the following questions.

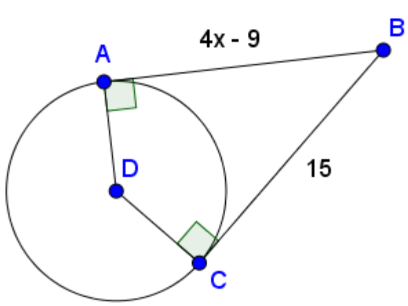
1. Is $\overleftrightarrow{AB}$ tangent to $\odot O$?



2. If $\overline{QP}$ is tangent to $\odot O$ at P, what is the radius of $\odot O$?



3. If $\overline{AB}$ and $\overline{CB}$ are tangents to $\odot D$, find the value of x.



Lesson 3: Arcs and Central Angles
Teacher Overview

Introduction: This lesson introduces students to the ideas of central angles and arcs, and has them explore the following important postulate related to arcs. Students will work in pairs, using GeoGebra.

- Arc Addition Postulate: The measure of the arc formed by two adjacent arcs is the sum of the measures of the two arcs.

Pre-requisite Knowledge:

- The measure of an exterior angle of a triangle is equal to the sum of the measures of the remote interior angles of the triangle.
- The base angles of an isosceles triangle are congruent.

Allotted Time: 1 class period (40 minutes)

Materials:

- For Students
    - Student Handout (1 per student)
    - Technology device with internet capability (at least 1 per pair of students)
- For Teacher
    - Teacher Notes
    - Annotated Student Handout

Lesson Overview:

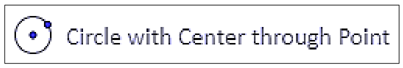
- Students work in pairs on Student Handout. (30 minutes)
    - Monitor student work by circulating around the classroom. If students are struggling, ask pointed questions to get them back on track. Refrain from giving students direct answers, as this is meant to be a discovery activity.
    - Once their figures have been checked, students can work on the sample problems to practice the concepts they learned.
- Bring students back into a whole group setting and go over the sample problems. Then assign homework problems related to the lesson content. (10 minutes)

Lesson 3: Arcs and Central Angles
Annotated Student Document

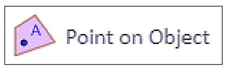
Today, we will introduce more circle vocabulary and visualize the terms using GeoGebra.

Part 1

☐ Go to www.geogebra.org/classic. Be sure to turn off the grid and axes features to declutter your screen.

☐ Using the 

tool, create circle *c* with center *A* and point *E* on the circle. Then hide *E*.

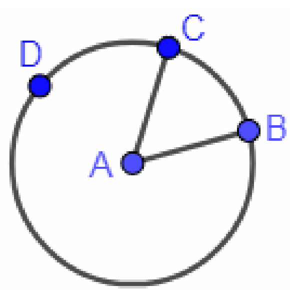
☐ Using the 

tool, put points *B*, *C* and *D* on *c*. All three points should be in the same half circle, with *C* between *B* and *D*.

☐ Using the 

tool, connect *A* and *B* to form $\overline{AB}$ and connect *A* and *C* to form $\overline{AC}$.

At this point, your figure should look similar to this, although your points may be elsewhere on the circle.



Take a look at $\angle BAC$. What are the sides of this angle? What is the vertex of this angle?

The sides of the angle are radii $\overline{AB}$ and $\overline{AC}$, and the vertex is the center of the circle, point *A*.

We call $\angle BAC$ a **central angle** of $\odot A$. Any central angle of a circle is an angle whose vertex is at the center of the circle.

An **arc** of a circle is a connected portion of a circle, with two endpoints that lie on the circle.

In Figure 1, notice that B and C are actually the endpoints of two different arcs, one starting at C and traveling clockwise to B and one starting at B and traveling clockwise to C (this is the dotted arc in Figure 1 below).

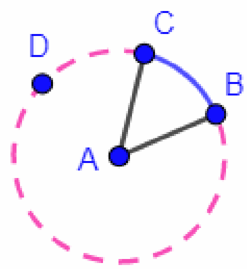


Figure 1

Arcs are classified in three ways:

- A **minor arc** is less than half of a circle.
  - To measure a minor arc, measure its central angle. The measure of a minor arc is between $0°$ and $180°$.
  - A minor arc is labeled using the two letters of its endpoints (i.e. $\overset{\frown}{BC}$ in Figure 1).

- A **major arc** is more than half of a circle.
  - The measure of a major arc is between $180°$ and $360°$. There are two ways to determine the measure of a major arc.
    - Subtract the measure of the minor arc formed by the same endpoints from $360°$.
    - Use GeoGebra to measure the central angle in the opposite direction from the measurement for the minor arc. For example, if we measure the central angle for the minor arc in the clockwise direction, then we measure the central angle for the major arc in the counterclockwise direction.
  - A major arc is labeled using the two letters of its endpoints plus a letter in between that represents a point on the circle between the endpoints (i.e. $\overset{\frown}{BDC}$ in Figure 1). Since two points on a circle define both a minor arc and a major arc, you may also see this notation used for a minor arc as a way of distinguishing between the two arcs.

- A **semicircle** is exactly half of a circle.
  - The endpoints of a semicircle lie on a diameter of the circle. A semicircle measures $180°$.
  - A semicircle is labeled in the same manner as a major arc.

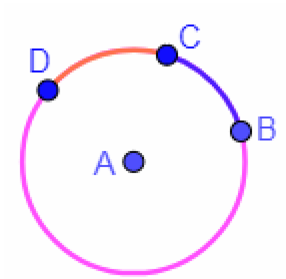
Congruent arcs are arcs that have equal measures in circles of equal radius.

Note: The Distance or Length tool measures the straight line distance between two points. Because we measure arcs in degrees, which is not a distance, we cannot use this tool to find an arc's measurement. In order to measure an arc (as you will need to do in future lessons), measure the central angle that corresponds to the arc using the Angle tool.

To practice, use the Angle tool to measure $\angle BAC$ both clockwise and counterclockwise. Record the measurements of the corresponding arcs below.

$m\overset{\frown}{BC} =$ _____ $m\overset{\frown}{BDC} =$ _____

Using the same circle, hide radii $\overline{AB}$ and $\overline{AC}$. Consider $\overset{\frown}{BC}$ and $\overset{\frown}{CD}$, as shown below.



We call $\overset{\frown}{BC}$ and $\overset{\frown}{CD}$ **adjacent arcs**, which are arcs that have one or two common endpoints, but don't have any other points in common. Just as the word *adjacent* means *next to*, adjacent arcs lie next to each other on a circle but do not overlap except at the endpoints.

☐ Using the Angle tool, find $m\angle CAD$ and $m\angle BAC$. Record the measurements below.

$m\overset{\frown}{CD} = m\angle CAD =$ _____ $m\overset{\frown}{BC} = m\angle BAC =$ _____

Notice that $\overset{\frown}{BD}$ is comprised of the adjacent arcs $\overset{\frown}{BC}$ and $\overset{\frown}{CD}$. Now find $m\angle BAD$. Record the measurement below.

$m\overset{\frown}{BD} = m\angle BAD =$ _____

What do you notice about mBD in comparison to the measures of the adjacent arcs BC and CD?

Students should observe that mBD is the sum of the measures of the adjacent arcs BC and CD.

☐ Use the 

tool to move *B*, *C*, and *D* to new locations on *c* so that one of the two arcs is a major arc. Does your observation still hold, even if the two adjacent minor arcs form a major arc? What does this suggest about the measure of an arc formed by two adjacent arcs? Fill in the conjecture statement below:

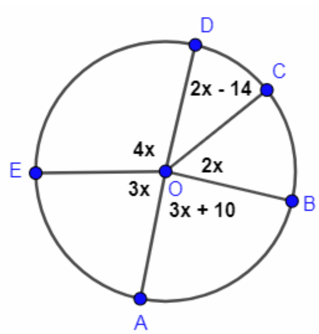
Students should note that their observation still holds; mBD is the sum of the measures of the adjacent arcs BC and CD.

Arc Addition Postulate: The measure of the arc formed by two adjacent arcs is the sum of the measures of these two arcs.

Have your teacher check your conjecture before moving on!

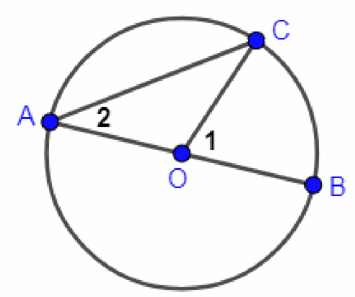
Use the ideas from this section to answer the following questions.

1. The measures of the central angles in the figure below are given in terms of *x*. Find *x* and use it to find $mAB,\ mBC,\ mCD,\ mDE,$ and mEA in $\odot O$.



Together, these arcs all form a full circle so their sum must be $360°$. By the Arc Addition Postulate, $(3x+10)+2x+(2x-14)+4x+3x=360$. Thus, $x=26$. So $mAB=88°$, $mBC=52°$, $mCD=38°$, $mDE=104°$, and $mEA=78°$.

2. In the figure, $\overline{AB}$ is a diameter of $\odot O$. Use the figure to fill in the missing information in the table.



	Ex. 1	Ex. 2	Ex. 3	Ex. 4
$m\overset{\frown}{CB}$	60°	58°	62°	$2x°$
$m\angle 1$	60°	58°	62°	$2x°$
$m\angle 2$	30°	29°	31°	$x°$

In the figure, $m\overset{\frown}{CB}$ is always equal to $m\angle 1$ because the measure of a minor arc is equal to the measure of its central angle and vice versa. The triangle ΔAOC is isosceles because two of its sides, $\overline{OA}$ and $\overline{OC}$ are congruent radii of $\odot O$. Notice also that $\angle 1$ is an exterior angle of ΔAOC so its measure is equal to the sum of the two remote interior angles. These angles are the base angles of the isosceles triangle so they are congruent. Therefore, $2(m\angle 2) = m\angle 1$ and $m\angle 2 = \frac{1}{2}(m\angle 1)$.

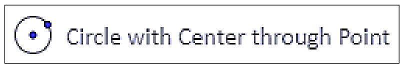
Note: In completing this example, students demonstrated that an inscribed angle is half the measure of its intercepted arc. They should be able to generalize their work to construct a proof of this result. We will revisit this result in Lesson 5.

Lesson 3: Arcs and Central Angles
Student Document

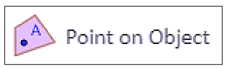
Today, we will introduce more circle vocabulary and visualize the terms using GeoGebra.

Part 1

☐ Go to www.geogebra.org/classic. Be sure to turn off the grid and axes features to declutter your screen.

☐ Using the 

tool, create circle *c* with center *A* and point *E* on the circle. Then hide *E*.

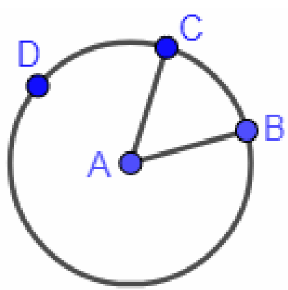
☐ Using the 

tool, put points *B*, *C* and *D* on *c*. All three points should be in the same half circle, with *C* between *B* and *D*.

☐ Using the 

tool, connect *A* and *B* to form $\overline{AB}$ and connect *A* and *C* to form $\overline{AC}$.

At this point, your figure should look similar to this, although your points may be elsewhere on the circle.



Take a look at $\angle BAC$. What are the sides of this angle? What is the vertex of this angle?

We call $\angle BAC$ a **central angle** of $\odot A$. Any central angle of a circle is an angle whose vertex is at the center of the circle.

An **arc** of a circle is a connected portion of a circle, with two endpoints that lie on the circle.

In Figure 1, notice that B and C are actually the endpoints of two different arcs, one starting at C and traveling clockwise to B and one starting at B and traveling clockwise to C (this is the dotted arc in Figure 1 below).

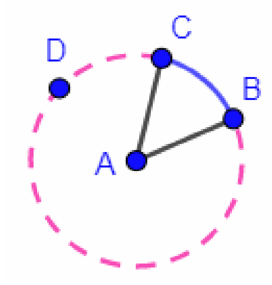


Figure 1

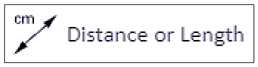
Arcs are classified in three ways:

- A **minor arc** is less than half of a circle.
    - To measure a minor arc, measure its central angle. The measure of a minor arc is between $0°$ and $180°$.
    - A minor arc is labeled using the two letters of its endpoints (i.e. BC in Figure 1).

- A **major arc** is more than half of a circle.
    - The measure of a major arc is between $180°$ and $360°$. There are two ways to determine the measure of a major arc.
        - Subtract the measure of the minor arc formed by the same endpoints from $360°$.
        - Use GeoGebra to measure the central angle in the opposite direction from the measurement for the minor arc. For example, if we measure the central angle for the minor arc in the clockwise direction, then we measure the central angle for the major arc in the counterclockwise direction.
    - A major arc is labeled using the two letters of its endpoints plus a letter in between that represents a point on the circle between the endpoints (i.e. BDC in Figure 1). Since two points on a circle define both a minor arc and a major arc, you may also see this notation used for a minor arc as a way of distinguishing between the two arcs.

- A **semicircle** is exactly half of a circle.
    - The endpoints of a semicircle lie on a diameter of the circle. A semicircle measures $180°$.
    - A semicircle is labeled in the same manner as a major arc.

Congruent arcs are arcs that have equal measures in circles of equal radius.

Note: The 

tool measures the straight line distance between two points. Because we measure arcs in degrees, which is not a distance, we cannot use this tool to find an arc’s measurement. In order to measure an arc (as you will need to do in future lessons), measure the central angle that corresponds to the arc using the 

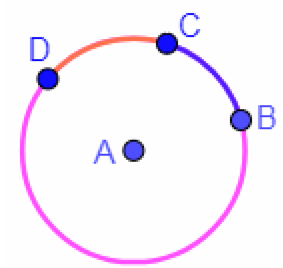
tool.

To practice, use the 

tool to measure $\angle BAC$ both clockwise and counterclockwise. Record the measurements of the corresponding arcs below.

$mBC =$ _____ $mBDC =$ _____

Using the same circle, hide radii $\overline{AB}$ and $\overline{AC}$. Consider BC and CD, as shown below.



We call BC and CD **adjacent arcs**, which are arcs that have one or two common endpoints, but don’t have any other points in common. Just as the word *adjacent* means *next to*, adjacent arcs lie next to each other on a circle but do not overlap except at the endpoints.

☐ Using the 

tool, find $m\angle CAD$ and $m\angle BAC$. Record the measurements below.

$mCD = m\angle CAD =$ _____ $mBC = m\angle BAC =$ _____

Notice that BD is comprised of the adjacent arcs BC and CD. Now find $m\angle BAD$. Record the measurement below.

$mBD = m\angle BAD =$ _____

What do you notice about $m\overset{\frown}{BD}$ in comparison to the measures of the adjacent arcs $\overset{\frown}{BC}$ and $\overset{\frown}{CD}$?

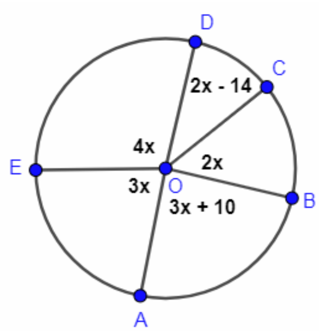
☐ Use the Move tool to move *B*, *C*, and *D* to new locations on *c* so that one of the two arcs is a major arc. Does your observation still hold, even if the two adjacent minor arcs form a major arc? What does this suggest about the measure of an arc formed by two adjacent arcs? Fill in the conjecture statement below:

Arc Addition Postulate: The measure of the arc formed by two adjacent arcs is the __________ of the measures of these two arcs.

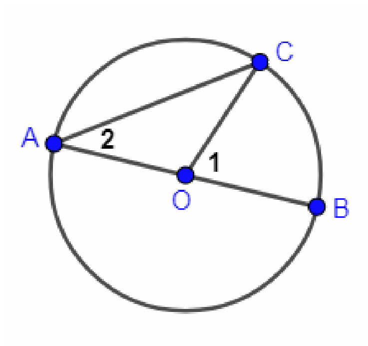
STOP Have your teacher check your conjecture before moving on!

Use the ideas from this section to answer the following questions.

1. The measures of the central angles in the figure below are given in terms of *x*. Find *x* and use it to find $m\overset{\frown}{AB}$, $m\overset{\frown}{BC}$, $m\overset{\frown}{CD}$, $m\overset{\frown}{DE}$, and $m\overset{\frown}{EA}$ in $\odot O$.



2. In the figure, $\overline{AB}$ is a diameter of $\odot O$. Use the figure to fill in the missing information in the table.



	Ex. 1	Ex. 2	Ex. 3	Ex. 4
mCB	60°			$2x$°
$m\angle 1$		58°		
$m\angle 2$			31°	

Lesson 4: Arcs and Chords
Teacher Overview

Introduction: This lesson allows students to explore theorems relating to chords and their arcs. Students will work in pairs using GeoGebra. The following theorems are covered.

- In a circle, congruent arcs have congruent chords, and vice versa.
- A diameter that is perpendicular to a chord bisects the chord and its arc.
- In a circle, chords that are equally distant from the center are congruent, and vice versa.

Pre-requisite Knowledge:

- Definitions of *congruent chords*, *congruent arcs*, and *bisector*
- Distance from a point to a line
- Side-Angle-Side, Side-Side-Side, and Hypotenuse-Leg triangle congruence
- Pythagorean Theorem
- How to type Greek letters in GeoGebra

Allotted Time: 1 class period (40 minutes)

Materials:

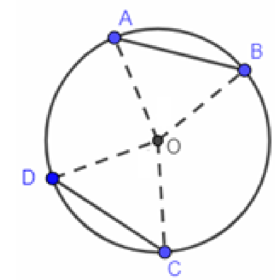
- For Students
  - Student Handout (1 per student)
  - Technology device with internet capability (at least 1 per pair of students)
- For Teacher
  - Teacher Notes and Annotated Student Handout

Lesson Overview:

- Students work in pairs on Student Handout. (25 minutes)
  - Monitor student work by circulating around the classroom. If students are struggling, ask pointed questions to get them back on track. Refrain from giving students direct answers, as this is meant to be a discovery activity.
  - As students form their conjectures, check to make sure the conjectures are accurate. If not, redirect students in a helpful manner to allow them to find the correct conclusions.
  - Once their conjectures have been checked, students can work on the sample problems to practice the concepts they learned.
- Bring students back into a whole group setting and go over the proofs of these conjectures. (10 minutes)
  - Use the proofs on the following page to lead the discussion, but ask for student input along the way.
  - Students should follow along and take notes on the proofs if necessary.
- Conclude the lesson by reviewing the sample problems and assigning homework problems related to the lesson content. (5 minutes)

Theorem: In a circle, congruent arcs have congruent chords, and congruent chords have congruent arcs.

Proof that congruent arcs have congruent chords:



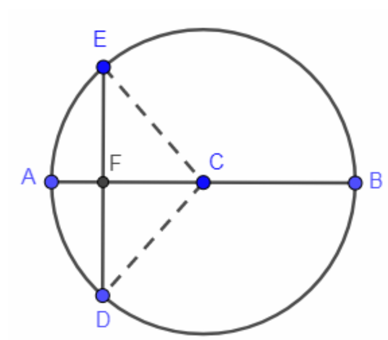
In $\odot O$, let $AB \cong CD$. Draw radii $\overline{OA}, \overline{OB}, \overline{OC},$ and $\overline{OD}$. Since all radii of a circle are congruent, $\overline{OA} \cong \overline{OD}$ and $\overline{OB} \cong \overline{OC}$. Since $AB \cong DC$, $\angle AOB \cong \angle DOC$. Now $\Delta OAB \cong \Delta ODC$ by Side-Angle-Side congruence and so $\overline{AB} \cong \overline{DC}$ because $\overline{AB}$ and $\overline{DC}$ are corresponding parts of the congruent triangles.

Note: In this illustration, AB and DC do not intersect. The proof is still valid if the arcs do intersect.

Proof that congruent chords have congruent arcs:

In $\odot O$, let $\overline{AB}$ and $\overline{CD}$ be congruent chords. Draw radii $\overline{OA}, \overline{OB}, \overline{OC},$ and $\overline{OD}$. Since all radii of a circle are congruent, $\overline{OA} \cong \overline{OD}$ and $\overline{OB} \cong \overline{OC}$. Now $\Delta OAB \cong \Delta ODC$ by Side-Side-Side congruence. Then $\angle AOB \cong \angle DOC$, and thus $AB \cong DC$ because arcs have the same measures as their central angles.

Theorem: A diameter that is perpendicular to a chord bisects the chord and its arc.



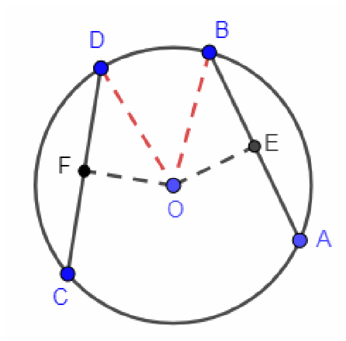
Proof: In $\odot C$, let $\overline{AB}$ be a diameter that is perpendicular to chord $\overline{ED}$. Let F be the intersection of $\overline{AB}$ and $\overline{ED}$. Draw radii $\overline{CE}$ and $\overline{CD}$. Since all radii of a circle are congruent, $\overline{CE} \cong \overline{CD}$. Also, $\overline{CF} \cong \overline{CF}$ so $\Delta CFE \cong \Delta CFD$ by Hypotenuse-Leg congruence. Therefore, $\overline{EF} \cong \overline{DF}$ and $\angle ECA \cong \angle DCA$. Thus $AE \cong AD$. So $\overline{AB}$ bisects chord $\overline{ED}$ and its arc ED.

Note: It's also true that a radius that is perpendicular to a chord bisects the chord and its arc, because a radius is part of a diameter.

Note: There is a straightforward extension of this proof that shows that a diameter also bisects the major arc formed by a chord.

Theorem: In a circle, chords that are equally distant from the center are congruent, and congruent chords are equally distant from the center.

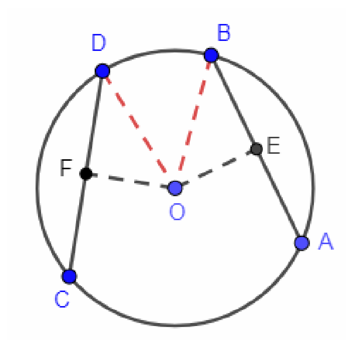
Proof that chords that are equally distant from the center are congruent:



In ⊙ O, let $\overline{AB}$ and $\overline{CD}$ be chords that are equally distant from the center *O*. This means that if we draw perpendicular segments from *O* to point *E* on $\overline{AB}$ and point *F* on $\overline{CD}$, then $\overline{OE} \cong \overline{OF}$. By the theorem above, *E* and *F* are midpoints of $\overline{AB}$ and $\overline{CD}$, respectively. Draw radii $\overline{OD}$ and $\overline{OB}$. Since all radii of a circle are congruent, $\overline{OD} \cong \overline{OB}$. Now $\Delta FOD \cong \Delta EOB$ by Hypotenuse-Leg congruence and so $\overline{FD} \cong \overline{EB}$. Since $2FD = CD$ and $2EB = AB$, $FD = EB$ implies $2FD = 2EB$. Therefore, $CD = AB$ and so $\overline{CD} \cong \overline{AB}$.

Note: In this illustration, $\overline{AB}$ and $\overline{CD}$ do not intersect. The proof is still valid if the chords do intersect.

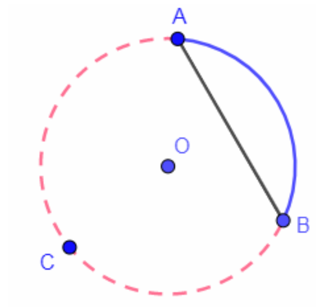
Proof that congruent chords are equally distant from the center:



In ⊙ O, let $\overline{AB}$ and $\overline{CD}$ be congruent chords. Construct $\overline{OE}$ so that *E* is on $\overline{AB}$ and $\overline{OE} \perp \overline{AB}$, and construct $\overline{OF}$ so that *F* is on $\overline{CD}$ and $\overline{OF} \perp \overline{CD}$. By the previous theorem, $\overline{OF}$ bisects $\overline{CD}$ and so $2FD = CD$. Similarly, $\overline{OE}$ bisects $\overline{AB}$ and thus $2EB = AB$. Since $AB = CD$, it follows that $2EB = 2FD$. Thus $EB = FD$ and $\overline{EB} \cong \overline{FD}$. Draw radii $\overline{OD}$ and $\overline{OB}$. Since all radii of a circle are congruent, $\overline{OD} \cong \overline{OB}$. Now $\Delta FOD \cong \Delta EOB$ by Hypotenuse-Leg congruence and so $\overline{OF} \cong \overline{OE}$. Thus, $\overline{AB}$ and $\overline{CD}$ are equally distant from the center *O*.

Lesson 4: Arcs and Chords
Annotated Student Document

Consider chord $\overline{AB}$ in $\odot O$. Notice that $\overline{AB}$ cuts off two arcs on $\odot O$.



- Minor arc $\overset{\frown}{AB}$
- Major arc $\overset{\frown}{ACB}$

The **arc of chord** $\overline{AB}$ refers to minor arc $\overset{\frown}{AB}$.

We will now explore a few theorems that relate chords and their arc measures.

Part 1

☐ Go to www.geogebra.org/classic. Be sure to turn off the grid and axes features to declutter your screen.

☐ Using the [Circle with Center through Point] tool, create circle *c* with center *O* and point *E* on the circle. Then hide *E*.

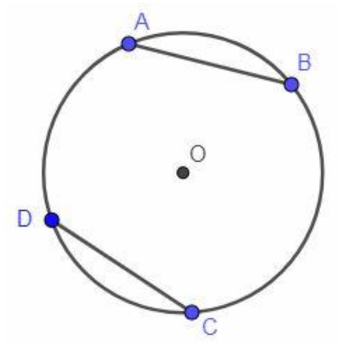
☐ Using the [Point on Object] tool, put points *A*, *B*, and *C* on *c*.

☐ Using the Segment tool, connect *A* and *B* to form $\overline{AB}$. GeoGebra calls both this segment and its length "*f*".

☐ Now we need to create a second chord, $\overline{CD}$, that has the exact same length as $\overline{AB}$. In order to ensure that the lengths of the chords are the same, use the [Circle with Center and Radius] tool to create a circle *d* with center at *C* and radius length *f* (the length of $\overline{AB}$).

☐ Recall that the circle *d* is all of the points that are *f* units away from *C*. Using the [Intersect] tool, find one of the two intersection points of *c* and *d*, and label the intersection *D*. Then use the Segment tool to connect *C* and *D* to form $\overline{CD}$, a chord that is congruent to $\overline{AB}$. Finally, hide circle *d*.

At this point, your figure should look similar to this, although your points may be elsewhere on the circle.



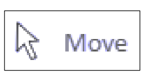
We will explore the measures of the arcs of the congruent chords $\overline{AB}$ and $\overline{CD}$.

☐ Using the  tool, find $mAB = m\angle AOB$ and $mCD = m\angle COD$. Record these measurements below.

$mAB =$ _____ $mCD =$ _____

Compare mAB and mCD. What do you notice?

Students should observe that the measures of the arcs are equal.

☐ Use the  tool to move *A*, *B*, and *C* to different locations on *c*, including locations where $\overline{AB}$ and $\overline{CD}$ intersect. Does your observation still hold? What does this imply about the measures of two arcs that are cut off by congruent chords? Fill in the conjecture statement appropriately:

Students should note that their observation still holds. The arcs have equal measures.

In a circle, congruent chords have congruent arcs.

Have your teacher check your conjecture before moving on!

What is the converse of this conjecture?

Congruent arcs have congruent chords.

Do you think the converse holds, too? Let's check it out.

- ☐ Open a new GeoGebra window.

- ☐ Using the Circle with Center through Point tool, create circle *c* with center *O* and point *E* on the circle. Then hide *E*.

- ☐ Using the Point on Object tool, put points *A*, *B*, and *C* on *c*.

- ☐ Using the Segment tool, connect *A* and *B* to form $\overline{AB}$.

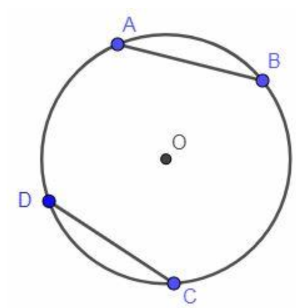
- ☐ Using the Angle tool, find the measure of central angle $\angle AOB$. Notice that GeoGebra calls this angle α .

- ☐ Now we need to create a second central angle, $\angle COD$, that has the exact same measure as $\angle AOB$. In order to ensure that the angle measurements are the same, use the Angle with Given Size tool to create $\angle COD$ with measure α .

- ☐ Using the Segment tool, connect *C* and the newly created *D* to form $\overline{CD}$.

  Because $\angle AOB$ and $\angle COD$ have the same measures, their arcs AB and CD will have the same measures.

  At this point, your figure should look similar to this, although your points may be elsewhere on the circle.



Now we can explore the measures of the chords $\overline{AB}$ and $\overline{CD}$ which have congruent arcs.

☐ Using the Distance or Length tool, measure chords $\overline{AB}$ and $\overline{CD}$. Record the measurements below.

$AB =$ _____ $CD =$ _____

Compare AB and CD. What do you notice? Does the converse to the last conjecture appear to be true? Fill in a new conjecture statement appropriately:

Students should observe that the measures of the chords are equal. Yes, the converse statement is true.

In a circle, congruent arcs have congruent chords.

STOP Have your teacher check your conjecture before moving on!

Part 2

☐ Open a new GeoGebra window.

☐ Using the Circle with Center through Point tool, create circle *c* with center *C* and point *G* on the circle. Then hide *G*.

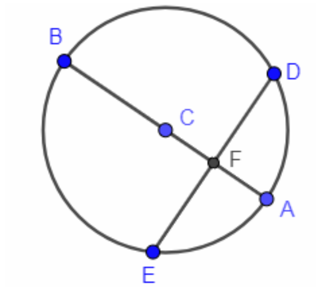
☐ Using the Point on Object tool, put points *D* and *E* on *c*. Then use the Segment tool to connect *D* and *E* to form chord $\overline{DE}$.

☐ Using the Perpendicular Line tool, create line *g* through *C* and perpendicular to $\overline{DE}$.

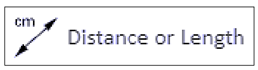
☐ Using the Intersect tool, find the intersections *A* and *B* of line *g* and circle *c*. Then use the Segment tool to connect *A* and *B* to form diameter $\overline{AB}$ and hide line *g*.

☐ Using the Intersect tool, find the intersection *F* of $\overline{AB}$ and $\overline{DE}$.

At this point, your figure should look similar to this, although your points may be elsewhere on the circle.



We will now explore the results of the perpendicular intersection of diameter $\overline{AB}$ and chord $\overline{DE}$.

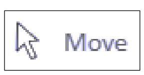
☐ Using the 

tool, measure the lengths of $\overline{EF}$, $\overline{FD}$, and $\overline{DE}$. Record the measurements below.

$EF =$ _____ $FD =$ _____ $DE =$ _____

Compare these three measurements. What do you notice?

Students should observe that $EF = FD = \frac{1}{2}DE$ or that $DE = 2EF = 2FD$.

☐ Use the 

tool to move *D* and *E* around *c*. Does your observation still hold? What does this imply about a diameter that is perpendicular to a chord? Fill in the conjecture statement appropriately:

Students should note that their observation still holds, $EF = FD = \frac{1}{2}DE$. This implies that *F* is the midpoint of $\overline{DE}$ and that diameter $\overline{AB}$ bisects chord $\overline{DE}$.

A diameter that is perpendicular to a chord bisects the chord.

☐ Using the 

tool, find $mAD = m\angle ACD$, $mAE = m\angle ACE$, and $mDE = m\angle DCE$. Record the measurements below.

$mAD =$ _____ $mAE =$ _____

$mDE =$ _____

Compare the measures of each of the three arcs. What do you notice?

Students should observe that $mAD = mAE = \frac{1}{2} mDE$ or that

$mDE = 2mAD = 2mAE$.

☐ Use the Move tool to move D and E around c. Does your observation still hold? What does this imply about a diameter that is perpendicular to a chord? Fill in the conjecture statement appropriately in relation to the arc of the chord:

Students should note that their observation still holds, $mAD = mAE = \frac{1}{2} mDE$.

This implies that diameter $\overline{AB}$ bisects DE , the arc of chord $\overline{DE}$.

A diameter that is perpendicular to a chord bisects the arc of the chord.

STOP Have your teacher check your conjectures before moving on!

Part 3

☐ Open a new GeoGebra window.

☐ Using the Circle with Center through Point tool, create circle c with center O and point G on the circle. Then hide G.

☐ Using the Point on Object tool, put points A and B on c.

☐ Using the Segment tool, connect A and B to form chord $\overline{AB}$.

☐ Using the Midpoint or Center tool, find the midpoint of $\overline{AB}$ and label it E.

☐ Using the Perpendicular Line tool, create line g through O and perpendicular to $\overline{AB}$.

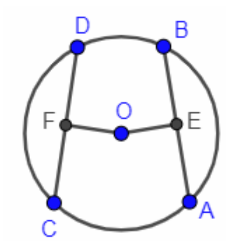
☐ Then use the Segment tool to connect O and E to form $\overline{OE}$, which GeoGebra calls h. Hide line g.

☐ Using the Segment with Given Length tool, create $\overline{OF}$ with length h. Then move F to a location of your choosing.

☐ Using the Perpendicular Line tool, create line j through F and perpendicular to $\overline{OF}$.

☐ Using the Intersect tool, find the intersections C and D of line j and circle c. Then use the Segment tool to connect C and D to form $\overline{CD}$. Hide line j.

The way we constructed $\overline{CD}$ ensures that $\overline{AB}$ and $\overline{CD}$ are equally distant from the center O. At this point, your figure should look similar to this, although your points may be elsewhere on the circle.



☐ Using the Distance or Length tool, measure $\overline{AB}$ and $\overline{CD}$. Record the measurements below.

$AB =$ ______ $CD =$ ______

Compare these two measurements. What do you notice?

Students should observe that the chords' measures are equal.

☐ Use the Move tool to move points A and B around c, including to some locations where $\overline{AB}$ and $\overline{CD}$ intersect. Does your observation hold? What does this imply about two chords that are equally distant from the center of a circle? Fill in the conjecture statement appropriately:

Students should note that the observation still holds. The chords' measures are equal.

In a circle, chords that are equally distant from the center are congruent.

STOP Have your teacher check your conjecture before moving on!

What is the converse of this conjecture?

Congruent chords are equally distant from the center of a circle.

Do you think the converse holds, too? Let's check it out.

☐ Open a new GeoGebra window.

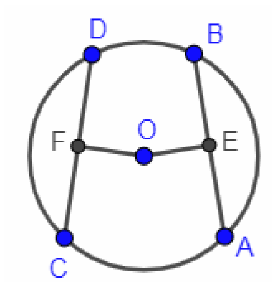
☐ Create congruent chords $\overline{AB}$ and $\overline{CD}$ in circle *O* as you did in Part 1.

To determine whether these chords are equally distant from *O*, we need to measure the perpendicular segments from *O* to chords $\overline{AB}$ and $\overline{CD}$. Recall from a previous conjecture in this lesson that a diameter or radius that is perpendicular to a chord bisects the chord.

☐ Using the [Midpoint or Center] tool, find the midpoints *E* and *F* of $\overline{AB}$ and $\overline{CD}$, respectively.

☐ Using the Segment tool, connect *O* and *E* to form $\overline{OE}$, and connect *O* and *F* to form $\overline{OF}$.

At this point, your figure should look similar to this, although your points may be elsewhere on the circle.



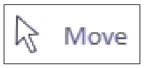
☐ Using the [Distance or Length] tool, measure $\overline{OE}$ and $\overline{OF}$. Record the measurements below.

$OE =$ ______ , which is the distance from O to $\overline{AB}$

$OF =$ ______ , which is the distance from O to $\overline{CD}$

Compare these two measurements. What do you notice?

Students should observe that the measures are equal.

☐ Use the 

tool to move points A and B around c, including to some locations where $\overline{AB}$ and $\overline{CD}$ intersect. Does your observation hold? What does this imply about two congruent chords and their distance from the center of a circle? Fill in the conjecture statement appropriately:

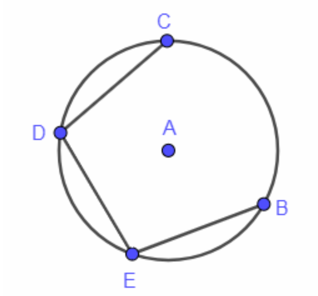
Students should note that the observation still holds. The measures are equal.

In a circle, congruent chords are equally distant from the center.

Have your teacher check your conjecture before moving on!

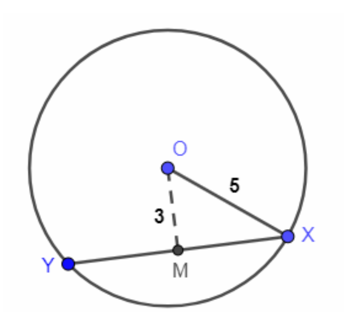
Use your conjectures to answer the following questions.

1. If $\overline{CD} \cong \overline{DE} \cong \overline{EB}$ and $mCB = 120°$, what is mCD?



If $mCB = 120°$, then there are $360° - 120° = 240°$ left for the other three arcs. Since $\overline{CD} \cong \overline{DE} \cong \overline{EB}$, their arcs are congruent so $3mCD = 240°$ and $mCD = 80°$.

2. Sketch a circle O with radius 5 cm and a chord $\overline{XY}$ that is 3 cm from O. How long is chord $\overline{XY}$?

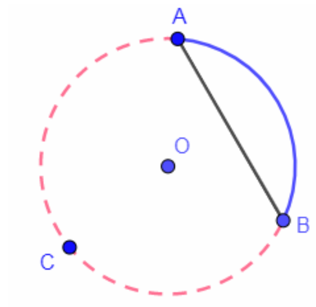


The diagram to the left is an accurate sketch of the scenario. Using the Pythagorean Theorem, $3^2 + (XM)^2 = 5^2$ and so $XM = 4$. Since $\overline{OM} \perp \overline{XY}$, $\overline{OM}$ bisects $\overline{XY}$ so $XY = 2XM = 8$.

Lesson 4: Arcs and Chords
Student Document

Consider chord $\overline{AB}$ in $\odot O$. Notice that $\overline{AB}$ cuts off two arcs on $\odot O$.

- Minor arc AB
- Major arc ACB

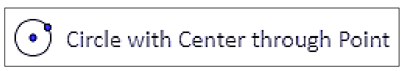


The **arc of chord** $\overline{AB}$ refers to minor arc AB.

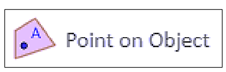
We will now explore a few theorems that relate chords and their arc measures.

Part 1

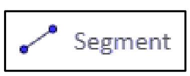
☐ Go to www.geogebra.org/classic. Be sure to turn off the grid and axes features to declutter your screen.

☐ Using the 

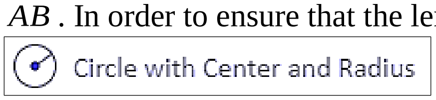
tool, create circle c with center O and point E on the circle. Then hide E.

☐ Using the 

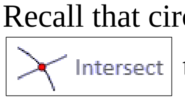
tool, put points A, B, and C on c.

☐ Using the 

tool, connect A and B to form $\overline{AB}$. GeoGebra calls both this segment and its length "f".

☐ Now we need to create a second chord, $\overline{CD}$, that has the exact same length as $\overline{AB}$. In order to ensure that the lengths of the chords are the same, use the 

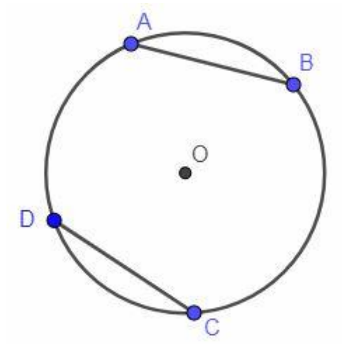
tool to create a circle d with center at C and radius length f (the length of $\overline{AB}$).

☐ Recall that circle d is all of the points that are f units away from C. Using the 

tool, find one of the two intersection points of c and d, and label the intersection D. Then use the 

tool to connect C and D to form $\overline{CD}$, a chord that is congruent to $\overline{AB}$. Finally, hide circle d.

At this point, your figure should look similar to this, although your points may be elsewhere on the circle.



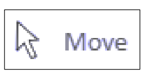
We will explore the measures of the arcs of the congruent chords $\overline{AB}$ and $\overline{CD}$.

☐ Using the 

tool, find $mAB = m\angle AOB$ and $mCD = m\angle COD$. Record these measurements below.

$mAB =$ _____ $mCD =$ _____

Compare mAB and mCD. What do you notice?

☐ Use the 

tool to move *A*, *B*, and *C* to different locations on *c*, including locations where $\overline{AB}$ and $\overline{CD}$ intersect. Does your observation still hold? What does this imply about the measures of two arcs that are cut off by congruent chords? Fill in the conjecture statement appropriately:

In a circle, congruent chords have _______________ arcs.

STOP Have your teacher check your conjecture before moving on!

What is the converse of this conjecture?

Do you think the converse holds, too? Let's check it out.

☐ Open a new GeoGebra window.

☐ Using the **Circle with Center through Point** tool, create circle c with center O and point E on the circle. Then hide E.

☐ Using the **Point on Object** tool, put points A, B, and C on c.

☐ Using the **Segment** tool, connect A and B to form $\overline{AB}$.

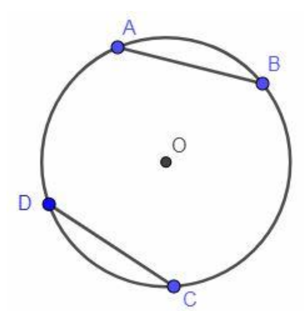
☐ Using the **Angle** tool, find the measure of central angle $\angle AOB$. Notice that GeoGebra calls this angle α.

☐ Now we need to create a second central angle, $\angle COD$, that has the exact same measure as $\angle AOB$. In order to ensure that the angle measurements are the same, use the **Angle with Given Size** tool to create $\angle COD$ with measure α.

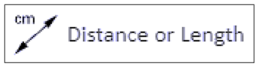
☐ Using the **Segment** tool, connect C and the newly created D to form $\overline{CD}$.

Because $\angle AOB$ and $\angle COD$ have the same measures, their arcs AB and CD will have the same measures.

At this point, your figure should look similar to this, although your points may be elsewhere on the circle.



Now we can explore the measures of the chords $\overline{AB}$ and $\overline{CD}$ which have congruent arcs.

☐ Using the 

tool, measure chords $\overline{AB}$ and $\overline{CD}$. Record the measurements below.

$AB =$ _____ $CD =$ _____

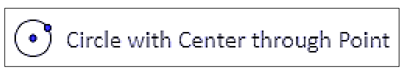
Compare AB and CD. What do you notice? Does the converse to the last conjecture appear to be true? Fill in a new conjecture statement appropriately:

In a circle, congruent arcs have _______________ chords.

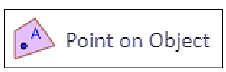
Have your teacher check your conjecture before moving on!

Part 2

☐ Open a new GeoGebra window.

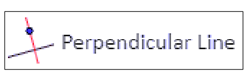
☐ Using the 

tool, create circle *c* with center *C* and point *G* on the circle. Then hide *G*.

☐ Using the 

tool, put points *D* and *E* on *c*. Then use the 

tool to connect *D* and *E* to form chord $\overline{DE}$.

☐ Using the 

tool, create line *g* through *C* and perpendicular to $\overline{DE}$.

☐ Using the 

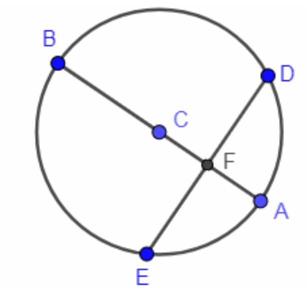
tool, find the intersections *A* and *B* of line *g* and circle *c*. Then use the 

tool to connect *A* and *B* to form diameter $\overline{AB}$ and hide line *g*.

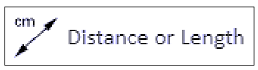
☐ Using the 

tool, find the intersection *F* of $\overline{AB}$ and $\overline{DE}$.

At this point, your figure should look similar to this, although your points may be elsewhere on the circle.



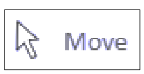
We will now explore the results of the perpendicular intersection of diameter $\overline{AB}$ and chord $\overline{DE}$.

☐ Using the 

tool, measure the lengths of $\overline{EF}$, $\overline{FD}$, and $\overline{DE}$. Record the measurements below.

$EF =$ _____ $FD =$ _____ $DE =$ _____

Compare these three measurements. What do you notice?

☐ Use the 

tool to move D and E around c. Does your observation still hold? What does this imply about a diameter that is perpendicular to a chord? Fill in the conjecture statement appropriately:

A diameter that is perpendicular to a chord ___________________________.

☐ Using the 

tool, find $mAD = m\angle ACD$, $mAE = m\angle ACE$, and $mDE = m\angle DCE$. Record the measurements below.

$mAD =$ _____ $mAE =$ _____

$mDE =$ _____

Compare the measures of each of the three arcs. What do you notice?

☐ Use the **Move** tool to move D and E around c. Does your observation still hold? What does this imply about a diameter that is perpendicular to a chord? Fill in the conjecture statement appropriately in relation to the arc of the chord:

A diameter that is perpendicular to a chord ____________________________.

STOP Have your teacher check your conjecture before moving on!

Part 3

☐ Open a new GeoGebra window.

☐ Using the **Circle with Center through Point** tool, create circle c with center O and point G on the circle. Then hide G.

☐ Using the **Point on Object** tool, put points A and B on c.

☐ Using the **Segment** tool, connect A and B to form chord $\overline{AB}$.

☐ Using the **Midpoint or Center** tool, find the midpoint of $\overline{AB}$ and label it E.

☐ Using the **Perpendicular Line** tool, create line g through O and perpendicular to $\overline{AB}$.

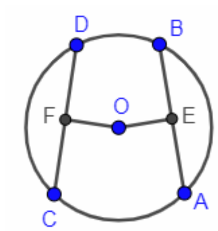
☐ Then use the **Segment** tool to connect O and E to form $\overline{OE}$, which GeoGebra calls h. Hide line g.

☐ Using the [Segment with Given Length] tool, create $\overline{OF}$ with length h. Then move F to a location of your choosing.

☐ Using the [Perpendicular Line] tool, create line j through F and perpendicular to $\overline{OF}$.

☐ Using the [Intersect] tool, find the intersections C and D of line j and circle c. Then use the Segment tool to connect C and D to form $\overline{CD}$. Hide line j.

The way we constructed $\overline{CD}$ ensures that $\overline{AB}$ and $\overline{CD}$ are equally distant from the center O. At this point, your figure should look similar to this, although your points may be elsewhere on the circle.



☐ Using the [Distance or Length] tool, measure $\overline{AB}$ and $\overline{CD}$. Record the measurements below.

$AB =$ _____ $CD =$ _____

Compare these two measurements. What do you notice?

☐ Use the [Move] tool to move points A and B around c, including some locations where $\overline{AB}$ and $\overline{CD}$ intersect. Does your observation hold? What does this imply about two chords that are equally distant from the center of a circle? Fill in the conjecture statement appropriately:

In a circle, chords that are equally distant from the center are __________.

STOP Have your teacher check your conjecture before moving on!

What is the converse of this conjecture?

Do you think the converse holds, too? Let's check it out.

☐ Open a new GeoGebra window.

☐ Create congruent chords $\overline{AB}$ and $\overline{CD}$ in circle O as you did in Part 1.

To determine whether these chords are equally distant from O, we need to measure the perpendicular segments from O to chords $\overline{AB}$ and $\overline{CD}$. Recall from a previous conjecture in this lesson that a diameter or radius that is perpendicular to a chord bisects the chord.

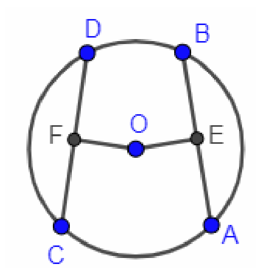
☐ Using the 

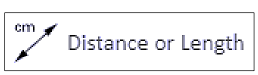
tool, find the midpoints E and F of $\overline{AB}$ and $\overline{CD}$, respectively.

☐ Using the 

tool, connect O and E to form $\overline{OE}$, and connect O and F to form $\overline{OF}$.

At this point, your figure should look similar to this, although your points may be elsewhere on the circle.



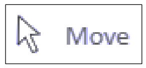
☐ Using the 

tool, measure $\overline{OE}$ and $\overline{OF}$. Record the measurements below.

$OE =$ _____ , which is the distance from _____ to _____

$OF =$ _____ , which is the distance from _____ to _____

Compare these two measurements. What do you notice?

☐ Use the 

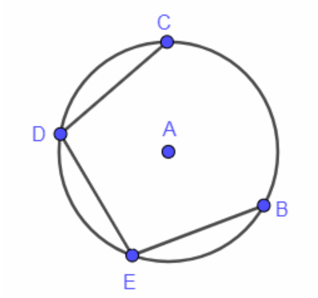
tool to move points A and B around c, including to some locations where $\overline{AB}$ and $\overline{CD}$ intersect. Does your observation hold? What does this imply about two congruent chords and their distance from the center of a circle? Fill in the conjecture statement appropriately:

In a circle, congruent chords are ____________________ from the center.

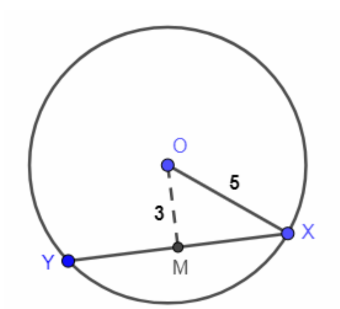
Have your teacher check your conjecture before moving on!

Use your conjectures to answer the following questions.

1. If $\overline{CD} \cong \overline{DE} \cong \overline{EB}$ and $mCB = 120°$, what is mCD?



2. Sketch a circle O with radius 5 cm and a chord $\overline{XY}$ that is 3 cm from O. How long is chord $\overline{XY}$?



Lesson 5: Inscribed Angles
Teacher Overview

Introduction: This lesson introduces students to inscribed angles and the relationship that inscribed angles have with their intercepted arcs. It also helps students to explore the following important theorems and corollaries related to inscribed angles. Students will work in pairs using GeoGebra.

Theorems:

- The measure of an inscribed angle is equal to half the measure of its intercepted arc.
- The measure of an angle formed by a chord and a tangent is equal to half the measure of the intercepted arc.

Corollaries:

- If two inscribed angles intercept the same arc, then the angles are congruent.
- An angle inscribed in a semicircle is a right angle.
- If a quadrilateral is inscribed in a circle, then its opposite angles are supplementary.

Pre-requisite Knowledge:

- The base angles of an isosceles triangle are congruent.
- The measure of an arc is equal to the measure of its central angle.
- The measure of an exterior angle of a triangle is equal to the sum of the measures of the remote interior angles of the triangle.
- Angle and Arc Addition Postulates
- An understanding of the definitions of *intercepted arcs* and *inscribed angles*

Allotted Time: 2 class periods (80 minutes)

Materials:

- For Students
  - Student Handout (1 per student)
  - Technology device with internet capability (at least 1 per pair of students)
- For Teacher
  - Teacher Notes
  - Annotated Student Handout

Lesson Overview:

Session 1

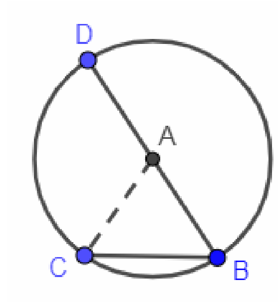
- Students work in pairs on Student Handout. (35 minutes)
  - Monitor student work by circulating around the classroom. If students are struggling, ask pointed questions to get them back on track. Refrain from giving students direct answers, as this is meant to be a discovery activity.
  - As students form their conjectures, check to make sure the conjectures are accurate. If not, redirect students in a helpful manner to allow them to find the correct conclusions.
- Conclude the lesson by bringing the students back into a whole group setting and assigning the sample problems for homework. (5 minutes)

Session 2

- Review the sample problems that students completed for homework. (10 minutes)
- Go over the proofs of the conjectures. (25 minutes)
  - Use the proofs below and on the following page to lead the discussion, but ask for student input along the way.
  - Students should follow along and take notes on the proofs if necessary.
- Conclude the lesson by assigning homework problems related to the lesson content. (5 minutes)

Theorem: The measure of an inscribed angle is equal to half the measure of its intercepted arc.

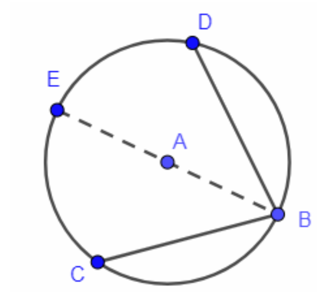
Proof: Let $\angle DBC$ be inscribed in $\odot A$.



Case 1: Assume one leg of $\angle DBC$ goes through *A*. Relabel points *C* and *D* if necessary so that *A* is on $\overline{DB}$. Draw radius $\overline{AC}$. Then $\overline{AC} \cong \overline{AB}$ because radii of a circle are congruent. Note that ΔABC is isosceles so its base angles are congruent and thus $m\angle ACB = m\angle ABC$. Since $\angle DAC$ is a central angle, its measure is equivalent to the measure of DC. Also, $\angle DAC$ is an exterior angle of ΔABC so its measure is the sum of the measures of the remote interior angles $\angle ACB$ and $\angle ABC$. Then, we have

$$\begin{aligned} mDC &= m\angle DAC \\ &= m\angle ACB + m\angle ABC \\ &= 2m\angle ABC. \end{aligned}$$

So $mDC = 2m\angle ABC$ and thus $\frac{1}{2}mDC = m\angle ABC$.

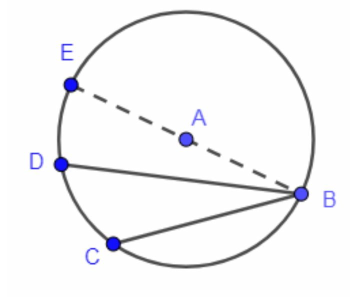


Case 2: Assume *A* is interior to $\angle DBC$. Draw a diameter with *B* as an endpoint. Label the other endpoint *E*. By Case 1 above, $m\angle CBE = \frac{1}{2}mCE$ and $m\angle EBD = \frac{1}{2}mED$. Using the addition property of equality,

$$\begin{aligned} m\angle CBE + m\angle EBD &= \frac{1}{2}mCE + \frac{1}{2}mED \\ &= \frac{1}{2}\left(mCE + mED\right). \end{aligned}$$

Notice that $m\angle CBE + m\angle EBD = m\angle CBD$ by the Angle Addition Postulate and $mCE + mED = mCD$ by the Arc Addition Postulate. Therefore, $m\angle CBD = \frac{1}{2}mCED$.

Case 3: Assume A is exterior to $\angle DBC$. Draw a diameter with B as an endpoint. Label the other endpoint E. Relabel C and D if necessary, so that CDE is a minor arc. By Case 1 above, $m\angle CBE = \frac{1}{2} mCE$ and $m\angle DBE = \frac{1}{2} mDE$. Using the subtraction property of equality,

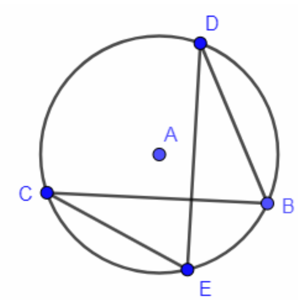


$$m\angle CBE - m\angle DBE = \frac{1}{2} mCE - \frac{1}{2} mDE$$
$$= \frac{1}{2}\left(mCE - mDE\right).$$

Notice that $m\angle CBE - m\angle DBE = m\angle CBD$ by the Angle Addition Postulate and $mCE - mDE = mCD$ by the Arc Addition Postulate. Therefore, $m\angle CBD = \frac{1}{2} mCD$.

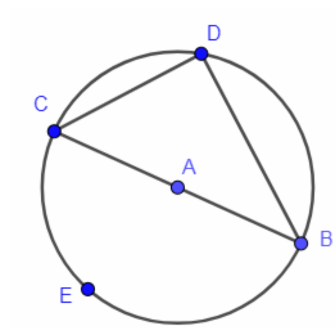
Corollary: If two inscribed angles intercept the same arc, then the angles are congruent.

Proof: Let B, C, D, and E be points on $\odot A$ such that $\angle CED$ and $\angle CBD$ are inscribed in $\odot A$ and they intercept the same arc. By the previous theorem, $m\angle CED = \frac{1}{2} mCD$ and $m\angle CBD = \frac{1}{2} mCD$. By substitution, $m\angle CED = m\angle CBD$.

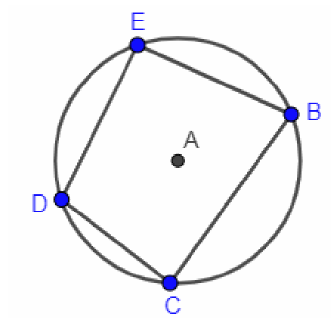


Corollary: An angle inscribed in a semicircle is a right angle.

Proof: Let $\angle CDB$ be inscribed in a semicircle with intercepted arc CEB. By the previous theorem, $m\angle CDB = \frac{1}{2} mCEB$. Since CEB is a semicircle, its measure is $180°$. By substitution, $m\angle CDB = \frac{1}{2}(180°) = 90°$. Therefore, $\angle CDB$ is a right angle.



Corollary: If a quadrilateral is inscribed in a circle, then its opposite angles are supplementary.

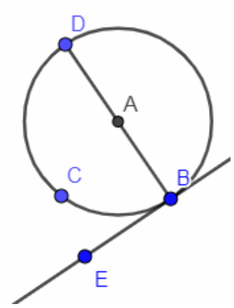


Proof: Let B, C, D, and E be points on $\odot A$ such that quadrilateral $BCDE$ is inscribed in $\odot A$. By the previous theorem, $m\angle EBC = \frac{1}{2} mEDC$ and $m\angle EDC = \frac{1}{2} mEBC$. By the addition property of equality,

$$\begin{aligned} m\angle EBC + m\angle EDC &= \frac{1}{2} mEDC + \frac{1}{2} mEBC \\ &= \frac{1}{2}\left(mEDC + mEBC\right). \end{aligned}$$

Notice that $mEDC + mEBC = 360°$. Then by substitution, $m\angle EBC + m\angle EDC = \frac{1}{2}(360°) = 180°$. Therefore, the opposite angles $\angle EBC$ and $\angle EDC$ are supplementary.

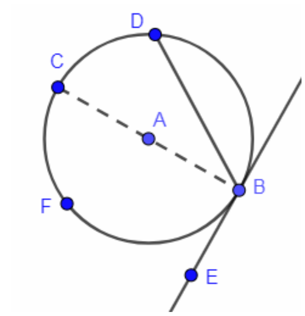
Theorem: The measure of an angle formed by a chord and a tangent is equal to half the measure of the intercepted arc.



Proof: Let $\overline{DB}$ be a chord in $\odot A$ and let $\overleftrightarrow{BE}$ be tangent to $\odot A$ at B.

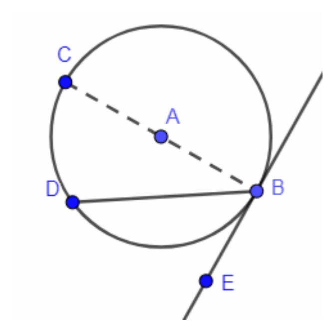
Case 1: Assume $\overline{DB}$ is a diameter and let C be a point on the circle in the interior of $\angle EBD$. Since $\overleftrightarrow{BE}$ is a tangent, $\overline{DB} \perp \overleftrightarrow{BE}$ by a theorem covered in Lesson 2 and thus $m\angle EBD = 90°$. Since $\overline{DB}$ is a diameter, DCB is a semicircle and $mDCB = 180°$. Therefore, $m\angle EBD = 90° = \frac{1}{2}(180°) = \frac{1}{2} mDCB$.

Case 2: Assume $\overline{DB}$ and $\overleftrightarrow{BE}$ intersect such that A is interior to $\angle EBD$. Draw a diameter with endpoint B. Label the other endpoint C. Since $\overleftrightarrow{BE}$ is a tangent, $\overline{CB} \perp \overleftrightarrow{BE}$ and $m\angle EBC = 90°$ by a theorem covered in Lesson 2. Let F be a point on the circle in the interior of $\angle EBC$. Since $\overline{CB}$ is a diameter, BFC is a semicircle and $mBFC = 180°$. Note that $\angle CBD$ is an inscribed angle, so its measure is half of the measure of its intercepted arc CD. Putting this information together,



$$\begin{aligned} m\angle EBD &= m\angle EBC + m\angle CBD \\ &= 90° + \frac{1}{2}mCD \\ &= \frac{1}{2}\left(180° + mCD\right) \\ &= \frac{1}{2}\left(mBFC + mCD\right) \\ &= \frac{1}{2}mBCD. \end{aligned}$$

Case 3: Assume $\overline{DB}$ and $\overleftrightarrow{BE}$ intersect such that A is exterior to $\angle EBD$. Draw a diameter with endpoint B. Label the other endpoint C. Since $\overleftrightarrow{BE}$ is a tangent, $\overline{CB} \perp \overleftrightarrow{EB}$ and $m\angle EBC = 90°$ by a theorem covered in Lesson 2. Since $\overline{CB}$ is a diameter, BDC is a semicircle and $mBDC = 180°$. Note that $\angle DBC$ is an inscribed angle, so its measure is half of the measure of its intercepted arc CD. Putting this information together,



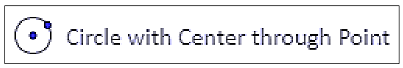
$$\begin{aligned} m\angle EBD &= m\angle EBC - m\angle DBC \\ &= 90° - \frac{1}{2}mCD \\ &= \frac{1}{2}\left(180° - mCD\right) \\ &= \frac{1}{2}\left(mBDC - mCD\right) \\ &= \frac{1}{2}mBD. \end{aligned}$$

Lesson 5: Inscribed Angles
Annotated Student Document

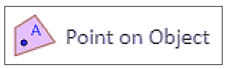
In this activity, we will explore some of the properties of inscribed angles and intercepted arcs.

Part 1

- ☐ Go to www.geogebra.org/classic. Be sure to turn off the grid and axes features to declutter your screen.

- ☐ Using the 

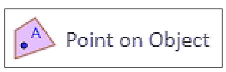
tool, create circle *c* with center *A* and point *G* on the circle. Then hide *G*.

- ☐ Using the 

tool, put points *B*, *C*, and *D* on *c*.

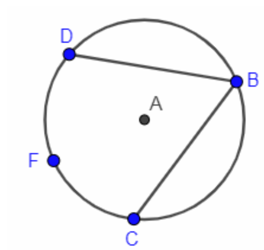
- ☐ Using the 

tool, connect *B* and *C* to form $\overline{BC}$ and connect *B* and *D* to form $\overline{BD}$.

- ☐ Using the 

tool, put point *F* on *c*, interior to $\angle CBD$. We will use *F* to label the intercepted arc of $\angle CBD$.

Your figure should look similar to this, although your points may be elsewhere on the circle, and *A* may be exterior to $\angle CBD$. Notice that $\angle CBD$ is an inscribed angle.



We will explore the measure of inscribed angle $\angle CBD$ in relation to the measure of its intercepted arc CFD.

- ☐ Using the 

tool, measure $\angle CBD$ and record its measure below.

$m\angle CBD =$ ______

☐ Using the Angle tool, find $mCFD = m\angle CAD$ and record its measure below.

$mCFD =$ _____

Compare $m\angle CBD$ and $mCFD$. What do you notice?

Students should notice that the measure of the inscribed angle is half the measure of the intercepted arc. If not, have them move on to the next step to gather more data and develop a pattern, perhaps using a table.

☐ Using the Move tool, move *B*, *C*, and *D* along *c*. Move *F* if necessary so that it is still interior to $\angle CBD$. Does your observation about $m\angle CBD$ and $mCFD$ still hold? What do your observations imply about the measure of an inscribed angle in relation to its intercepted arc? Fill in the conjecture statement appropriately:

Students should notice that their observation still holds. The measure of the inscribed angle is half the measure of the intercepted arc.

The measure of an inscribed angle is equal to half the measure of its intercepted arc.

STOP Have your teacher check your conjecture before moving on!

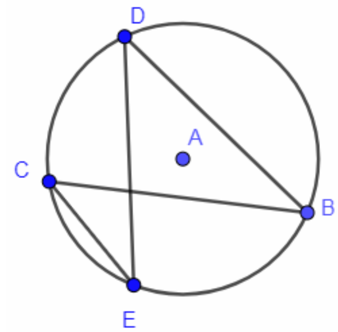
Part 2

☐ In your GeoGebra worksheet, hide the measurements that you found in Part 1.

☐ Using your figure from Part 1, use the Point on Object tool to create a point *E* on *c* such that *E* is exterior to $\angle CBD$. Hide point *F*.

☐ Using the Segment tool, connect *E* and *C* to form $\overline{EC}$ and connect *E* and *D* to form $\overline{ED}$.

Your figure should look similar to this, although your points may be elsewhere on the circle. Notice that both $\angle CBD$ and $\angle CED$ are inscribed angles which intercept CD.



What conjecture can you make regarding the measures of $\angle CBD$ and $\angle CED$ based on your work in Part 1?

Students should note that $m\angle CBD = m\angle CED$ based on the previous conjecture.

- ☐ Briefly describe how you can use GeoGebra to investigate this conjecture. Then investigate accordingly using various examples by moving the points around the circle. If your conjecture is wrong at any point, adjust it and try again. When you have tested several examples in which your conjecture holds, fill in the statement below:

Students can find the measures of inscribed angles $\angle CBD$ and $\angle CED$ to see if they are congruent.

If two inscribed angles intercept the same arc, then the angles are congruent.

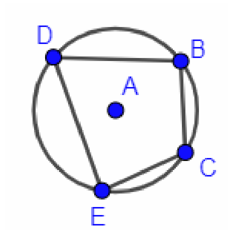
STOP Have your teacher check your conjecture before moving on!

Part 3

- ☐ In your GeoGebra worksheet, hide the measurements that you found in Part 2.

- ☐ Using your figure from Parts 1 and 2, use the Move tool to move points *B*, *C*, *D*, and *E* around *c* until you form a quadrilateral $BCED$ with *A* in the center.

Your figure should look similar to this, although your points may be elsewhere on the circle.



☐ Using the 

tool, measure each angle of the quadrilateral and record the measures below.

$m\angle B =$ ______ $m\angle E =$ ______

$m\angle C =$ ______ $m\angle D =$ ______

Notice that $\angle B$ and $\angle E$ are opposite angles of the quadrilateral, and that $\angle C$ and $\angle D$ are opposite angles of the quadrilateral. Add the measures of each pair of opposite angles together and record your sums below.

$m\angle B + m\angle E =$ _____ $m\angle C + m\angle D =$ ______

☐ Use the 

tool to move all four points around *c* to create a new inscribed quadrilateral *BCED*. Add up the measures of each pair of opposite angles once again. Does your observation still hold? What does this imply about the opposite angles of a quadrilateral inscribed in a circle? Fill in the conjecture statement appropriately:

Students should observe that opposite angles still add up to $180°$.

If a quadrilateral is inscribed in a circle, then its opposite angles are supplementary.

STOP Have your teacher check your conjecture before moving on!

Part 4

☐ Open a new GeoGebra window.

☐ Using the [Circle with Center through Point] tool, create circle c with center A and point E on the circle. Then hide E.

☐ Using the [Point on Object] tool, put point B on c.

☐ Using the [Line] tool, connect A and B to form $\overleftrightarrow{AB}$.

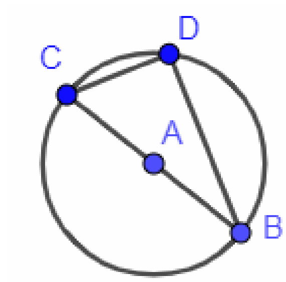
☐ Using the [Intersect] tool, find the intersection of c and $\overleftrightarrow{AB}$ on the opposite side of the circle from B. Label this point C.

☐ Using the **Segment** tool, connect B and C to form $\overline{BC}$, a diameter of the circle. Then hide $\overleftrightarrow{AB}$.

☐ Using the [Point on Object] tool, put point D on c.

☐ Using the **Segment** tool, connect C and D to form $\overline{CD}$ and connect B and D to form $\overline{BD}$.

Your figure should look similar to this, although your points may be elsewhere on the circle.



Notice that $\angle CDB$ is an angle inscribed in a semicircle. What do you think its measure will be, based on an earlier conjecture in this lesson?

Students should recognize that $m\angle CDB = \frac{1}{2}(180°) = 90°$.

☐ Using the **Angle** tool, measure $\angle CDB$ to see if you answered correctly.

$m\angle CDB = 90°$

☐ Use the **Move** tool to move B and D around c. Does your observation still hold? What does this imply about an angle inscribed in a semicircle? Fill in the conjecture statement appropriately:

Students should observe the angle measure is still $90°$.

An angle inscribed in a semicircle is a right angle.

STOP Have your teacher check your conjecture before moving on!

Part 5

☐ Open a new GeoGebra window.

☐ Using the **Circle with Center through Point** tool, create circle c with center A and point F on the circle. Then hide F.

☐ Using the **Point on Object** tool, put point B on circle c.

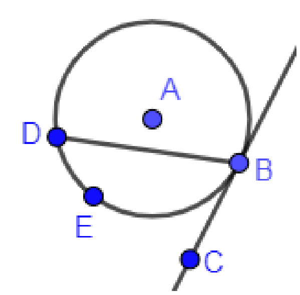
☐ Using the **Tangents** tool, create line f that is tangent to c and goes through B.

☐ Using the **Point on Object** tool, put point C on line f and point D on circle c.

☐ Using the **Segment** tool, connect D and B to form $\overline{DB}$.

☐ Using the **Point on Object** tool, put point E on c such that E is interior to $\angle CBD$.

Your figure should look similar to this.



What kind of segment and line are used to form $\angle CBD$?

$\angle CBD$ is formed by a chord and a tangent.

☐ Using the Angle tool, measure $\angle CBD$ and its intercepted arc BED. Record your measurements below.

$m\angle CBD =$ _____ $mBED =$ _____

What do you notice about these measurements?

Students should notice that the measure of the angle is half the measure of its intercepted arc. If not, have them move on to the next step to gather more data and develop a pattern, perhaps using a table.

☐ Use the Move tool to move *B* and *D* around *c*. Does your observation about

$m\angle CBD$ and $mBED$ still hold? What does this imply about the measure of an angle formed by a chord and a tangent in relation to its intercepted arc? Fill in the conjecture statement appropriately:

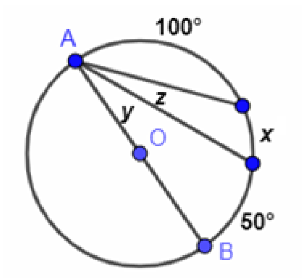
Students should notice that their observation still holds. The measure of the angle formed by a chord and a tangent is always half the measure of its intercepted arc.

The measure of an angle formed by a chord and a tangent is equal to half the measure of the intercepted arc.

STOP Have your teacher check your conjecture before moving on!

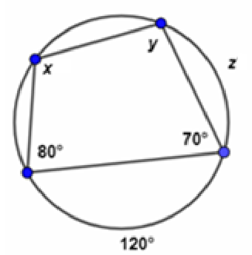
Use your conjectures to find the measures of *x*, *y*, and *z* in each figure.

1. In the figure below, $\overline{AB}$ is a diameter.



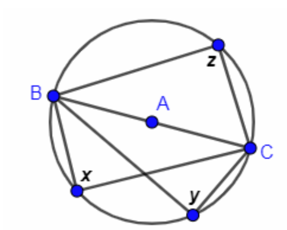
Diameter $\overline{AB}$ cuts the circle into two semicircles, so $100° + 50° + m\overset{\frown}{x} = 180°$ which results in $m\overset{\frown}{x} = 30°$. This is the measure of the intercepted arc for inscribed angle *z*, so $m\angle z = \frac{1}{2} m\overset{\frown}{x}$. This results in $m\angle z = 15°$. The intercepted arc for inscribed angle *y* measures $50°$, so the measure of *y* is $\frac{1}{2}(50°)$ which results in $m\angle y = 25°$.

2.



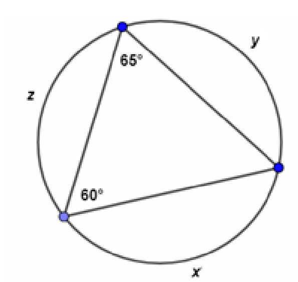
Opposite angles of an inscribed quadrilateral are supplementary so $m\angle x + 70° = 180°$ and thus $m\angle x = 110°$; and $m\angle y + 80° = 180°$, so $m\angle y = 100°$. Inscribed angle *x* is half the measure of its intercepted arc, so $110° = \frac{1}{2}\left(m\overset{\frown}{z} + 120°\right)$, which results in $m\overset{\frown}{z} = 100°$.

3. In the figure below, $\overline{BC}$ is a diameter.



Each angle is inscribed in a semicircle, so $x = y = z = 90°$.

4.



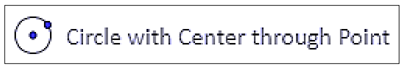
Because *x* and *y* are the intercepted arcs of inscribed angles, their measures are double those angles. Therefore, $m\overset{\frown}{x} = 2(65°) = 130°$ and $m\overset{\frown}{y} = 2(60°) = 120°$. The third angle of the triangle has a measure of $180° - (65° + 60°) = 55°$ and *z* is its intercepted arc. Then $m\overset{\frown}{z} = 2(55°) = 110°$.

Lesson 5: Inscribed Angles
Student Document

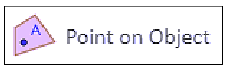
In this activity, we will explore some of the properties of inscribed angles and intercepted arcs.

Part 1

- ☐ Go to www.geogebra.org/classic. Be sure to turn off the grid and axes features to declutter your screen.

- ☐ Using the 

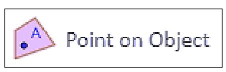
tool, create circle *c* with center *A* and point *G* on the circle. Then hide *G*.

- ☐ Using the 

tool, put points *B*, *C*, and *D* on *c*.

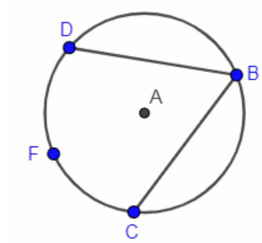
- ☐ Using the 

tool, connect *B* and *C* to form $\overline{BC}$ and connect *B* and *D* to form $\overline{BD}$.

- ☐ Using the 

tool, put point *F* on *c*, interior to $\angle CBD$. We will use *F* to label the intercepted arc of $\angle CBD$.

Your figure should look similar to this, although your points may be elsewhere on the circle, and *A* may be exterior to $\angle CBD$. Notice that $\angle CBD$ is an inscribed angle.



We will explore the measure of inscribed angle $\angle CBD$ in relation to the measure of its intercepted arc CFD.

- ☐ Using the 

tool, measure $\angle CBD$ and record its measure below.

$m\angle CBD =$ ______

☐ Using the Angle tool, find $mCFD = m\angle CAD$ and record its measure below.

$mCFD =$ _____

Compare $m\angle CBD$ and $mCFD$. What do you notice?

☐ Using the Move tool, move *B*, *C*, and *D* along *c*. Move *F* if necessary so that it is still interior to $\angle CBD$. Does your observation about $m\angle CBD$ and $mCFD$ still hold? What do your observations imply about the measure of an inscribed angle in relation to its intercepted arc? Fill in the conjecture statement appropriately:

The measure of an inscribed angle is _______________ the measure of its intercepted arc.

STOP Have your teacher check your conjecture before moving on!

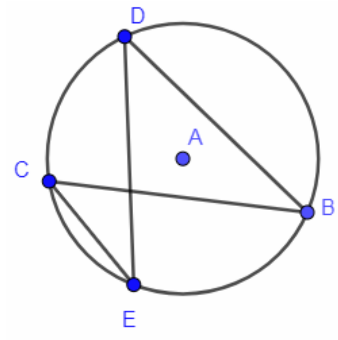
Part 2

☐ In your GeoGebra worksheet, hide the measurements that you found in Part 1.

☐ Using your figure from Part 1, use the Point on Object tool to create a point *E* on *c* such that *E* is exterior to $\angle CBD$.

☐ Using the Segment tool, connect *E* and *C* to form $\overline{EC}$ and connect *E* and *D* to form $\overline{ED}$.

Your figure should look similar to this, although your points may be elsewhere on the circle. Notice that both $\angle CBD$ and $\angle CED$ are inscribed angles which intercept CD.



What conjecture can you make regarding the measures of $\angle CBD$ and $\angle CED$ based on your work in Part 1?

☐ Briefly describe how you can use GeoGebra to investigate this conjecture. Then investigate accordingly using various examples by moving the points around the circle. If your conjecture is wrong at any point, adjust it and try again. When you have tested several examples in which your conjecture holds, fill in the statement below:

If two inscribed angles intercept the same arc, then the angles are _________.

Have your teacher check your conjecture before moving on!

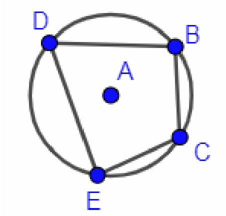
Part 3

☐ In your GeoGebra worksheet, hide the measurements that you found in Part 2.

☐ Using your figure from Parts 1 and 2, use the 

tool to move points *B*, *C*, *D*, and *E* around *c* until you form a quadrilateral $BCED$ with *A* in the center.

Your figure should look similar to this, although your points may be elsewhere on the circle.



☐ Using the 

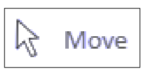
tool, measure each angle of the quadrilateral and record the measures below.

$m\angle B =$ _____ $m\angle E =$ _____

$m\angle C =$ _____ $m\angle D =$ _____

Notice that $\angle B$ and $\angle E$ are opposite angles of the quadrilateral, and that $\angle C$ and $\angle D$ are opposite angles of the quadrilateral. Add the measures of each pair of opposite angles together and record your sums below.

$m\angle B + m\angle E =$ _____ $m\angle C + m\angle D =$ _____

☐ Use the 

tool to move all four points around *c* to create a new inscribed quadrilateral *BCED*. Add up the measures of each pair of opposite angles once again. Does your observation still hold? What does this imply about the opposite angles of a quadrilateral inscribed in a circle? Fill in the conjecture statement appropriately:

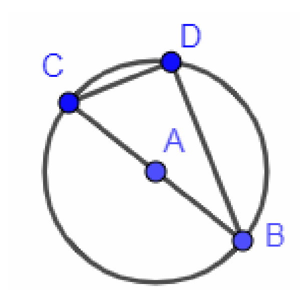
If a quadrilateral is inscribed in a circle, then its opposite angles are ____________________.

STOP Have your teacher check your conjecture before moving on!

Part 4

- ☐ Open a new GeoGebra window.
- ☐ Using the Circle with Center through Point tool, create circle *c* with center *A* and point *E* on the circle. Then hide *E*.
- ☐ Using the Point on Object tool, put point *B* on *c*.
- ☐ Using the Line tool, connect *A* and *B* to form $\overleftrightarrow{AB}$.
- ☐ Using the Intersect tool, find the intersection of *c* and $\overleftrightarrow{AB}$ on the opposite side of the circle from *B*. Label this point *C*.
- ☐ Using the Segment tool, connect *B* and *C* to form $\overline{BC}$, a diameter of the circle. Then hide $\overleftrightarrow{AB}$.
- ☐ Using the Point on Object tool, put point *D* on *c*.
- ☐ Using the Segment tool, connect *C* and *D* to form $\overline{CD}$ and connect *B* and *D* to form $\overline{BD}$.

Your figure should look similar to this, although your points may be elsewhere on the circle.



Notice that $\angle CDB$ is an angle inscribed in a semicircle. What do you think its measure will be based on an earlier conjecture in this lesson?

☐ Using the Angle tool, measure $\angle CDB$ to see if you answered correctly.

$m\angle CDB =$ _____

Then use the Move tool to move B and D around c. Does your observation still hold? What does this imply about an angle inscribed in a semicircle? Fill in the conjecture statement appropriately:

An angle inscribed in a semicircle is a __________ angle.

STOP Have your teacher check your conjecture before moving on!

Part 5

☐ Open a new GeoGebra window.

☐ Using the Circle with Center through Point tool, create circle c with center A and point F on the circle. Then hide F.

☐ Using the Point on Object tool, put point B on c.

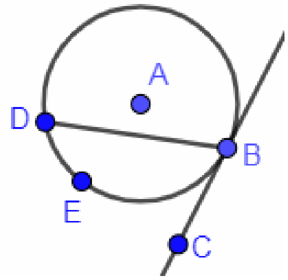
☐ Using the Tangents tool, create line f that is tangent to c and goes through B.

☐ Using the Point on Object tool, put point C on line f and point D on circle c.

☐ Using the Segment tool, connect D and B to form $\overline{DB}$.

☐ Using the Point on Object tool, put point E on c such that E is interior to $\angle CBD$.

Your figure should look similar to this.



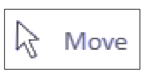
What kind of segment and line are used to form $\angle CBD$?

☐ Using the 

tool, measure $\angle CBD$ and its intercepted arc $\overset{\frown}{BED}$. Record your measurements below.

$m\angle CBD =$ _____ $m\overset{\frown}{BED} =$ _____

What do you notice about these measurements?

☐ Use the 

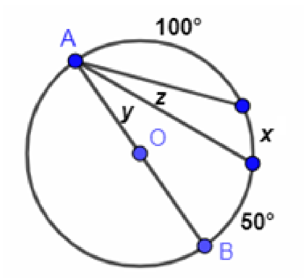
tool to move *B* and *D* around *c*. Does your observation about $m\angle CBD$ and $m\overset{\frown}{BED}$ still hold? What does this imply about the measure of an angle formed by a chord and a tangent in relation to its intercepted arc? Fill in the conjecture statement appropriately:

The measure of an angle formed by a chord and a tangent is ______________ the measure of the intercepted arc.

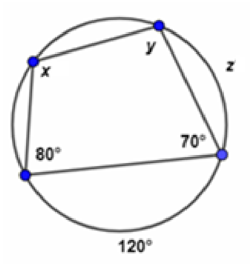
Have your teacher check your conjecture before moving on!

Use your conjectures to find the measures of *x*, *y*, and *z* in each figure.

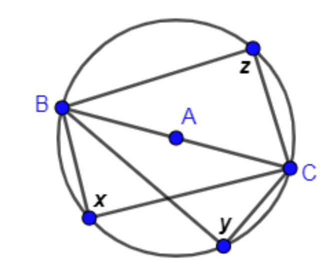
1. In the figure below, $\overline{AB}$ is a diameter.



2.



3. In the figure below, $\overline{BC}$ is a diameter.



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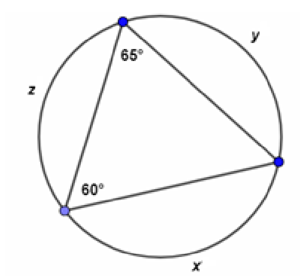


Lesson 6: Other Angles
Teacher Overview

Introduction: This lesson helps students to discover the measures of other important angles related to circles, including the angles formed by two secants, two tangents, or a secant and a tangent. Students will work in pairs, using GeoGebra. The following theorems are covered.

- The measure of an angle formed by two secants that intersect inside a circle is equal to half the sum of the measures of its intercepted arc and the intercepted arc of its corresponding vertical angle.
- The measure of an angle formed by any of the following is equal to half the absolute value of the difference of the measures of the intercepted arcs:
    - Two secants that intersect at a point outside a circle
    - Two tangents that intersect at a point outside a circle
    - A secant and a tangent

Pre-requisite Knowledge:

- The measure of an exterior angle of a triangle is equal to the sum of the measures of the remote interior angles of the triangle.
- The measure of an angle inscribed in a circle is equal to half the measure of its intercepted arc and the measure of an angle formed by a chord and a tangent is equal to half the measure of its intercepted arc.

Allotted Time: 1 class period (40 minutes)

Materials:

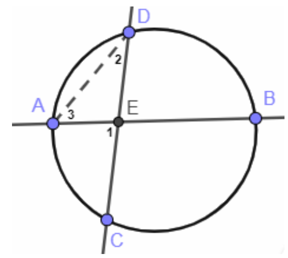
- For Students
    - Student Handout (1 per student)
    - Technology device with internet capability (at least 1 per pair of students)
- For Teacher
    - Teacher Notes
    - Annotated Student Handout

Lesson Overview:

- Students work in pairs on Student Handout. (20 minutes)
    - Monitor student work by circulating around the classroom. If students are struggling, ask pointed questions to get them back on track. Refrain from giving students direct answers, as this is meant to be a discovery activity.
    - As students form their conjectures, check to make sure the conjectures are accurate. If not, redirect students in a helpful manner to allow them to find the correct conclusions.
    - Once their conjectures have been checked, students can work on the sample problems to practice the concepts they learned.

- Bring students back into a whole group setting and go over the proofs of these conjectures. (15 minutes)
    - Use the proofs on the following page to lead the discussion, but ask for student input along the way.
    - Students should follow along and take notes on the proofs if necessary.
- Conclude the lesson by reviewing the sample problems and assigning homework problems related to the lesson content. (5 minutes)

Theorem: The measure of an angle formed by two secants that intersect inside a circle is equal to half the sum of the measures of its intercepted arc and the intercepted arc of its corresponding vertical angle.

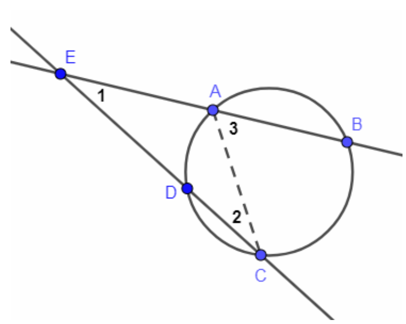


Proof: Let *A*, *B*, *C*, and *D* be points on the circle such that secants $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at point *E* inside the circle. Draw $\overline{AD}$. Let $\angle 1 = \angle AEC$, $\angle 2 = \angle ADE$, and $\angle 3 = \angle DAE$. Since $\angle 1$ is an exterior angle of ΔEAD, $m\angle 1 = m\angle 2 + m\angle 3$. Since $\angle 2$ and $\angle 3$ are inscribed angles of the circle, $m\angle 2 = \frac{1}{2} mAC$ and $m\angle 3 = \frac{1}{2} mDB$.

Then $m\angle 1 = \frac{1}{2} mAC + \frac{1}{2} mDB = \frac{1}{2}\left(mAC + mDB\right)$.

Note: In the illustration for this proof, the intercepted arcs AC and BD are minor arcs. The proof is still valid when either of those intercepted arcs is a major arc, though.

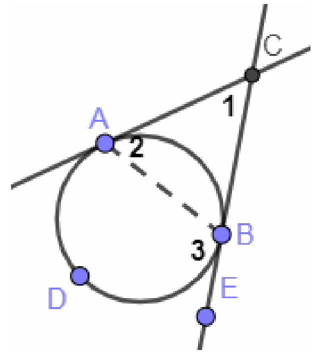
Theorem: The measure of an angle formed by two secants that intersect at a point outside a circle is equal to half the absolute value of the difference of the measures of the intercepted arcs.



Proof: Let *A*, *B*, *C*, and *D* be points on a circle such that secants $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$ intersect at point *E* outside of the circle. Relabel *A*, *B*, *C*, and *D* if necessary so that *A* is between *E* and *B*, and *D* is between *E* and *C*. Then the intercepted arcs of $\angle BEC$ are AD and BC. Draw $\overline{AC}$. Let $\angle 1 = \angle AED$, $\angle 2 = \angle DCA$, and $\angle 3 = \angle CAB$. Since $\angle 3$ is an exterior angle of ΔEAC, $m\angle 3 = m\angle 1 + m\angle 2$ and subsequently, $m\angle 1 = m\angle 3 - m\angle 2$. Since $\angle 2$ and $\angle 3$ are inscribed angles of the circle, $m\angle 2 = \frac{1}{2} mAD$ and $m\angle 3 = \frac{1}{2} mBC$. Then

$$\begin{aligned} m\angle 1 &= \left|\frac{1}{2} mBC - \frac{1}{2} mAD\right| \\ &= \frac{1}{2}\left|\left(mBC - mAD\right)\right|. \end{aligned}$$

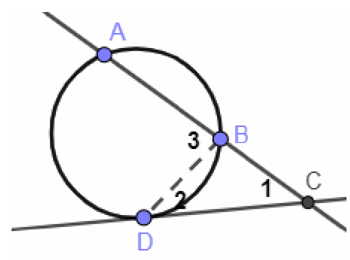
Theorem: The measure of an angle formed by two tangents that intersect at a point outside a circle is equal to half the absolute value of the difference of the measures of the intercepted arcs.



Proof: Let $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$ be tangents to a circle that intersect at C outside of the circle with points of tangency A and B. Draw $\overline{AB}$. Let $\angle 1 = \angle ACB$, $\angle 2 = \angle CAB$, and $\angle 3 = \angle ABE$, where E is a point on $\overleftrightarrow{BC}$ such that B is between C and E. Since $\angle 3$ is an exterior angle of ΔABC, $m\angle 3 = m\angle 1 + m\angle 2$ and subsequently, $m\angle 1 = m\angle 3 - m\angle 2$. Since $\angle 2$ and $\angle 3$ are angles formed by a tangent and a chord, we know from Lesson 5 that $m\angle 2 = \frac{1}{2} mAB$ and $m\angle 3 = \frac{1}{2} mADB$, where D is a point on the circle on the interior of $\angle ABE$. Then

$$\begin{aligned} m\angle 1 &= \left| \frac{1}{2} mADB - \frac{1}{2} mAB \right| \\ &= \frac{1}{2}\left| \left(mADB - mAB \right) \right|. \end{aligned}$$

Theorem: The measure of an angle formed by a secant and a tangent is equal to half the absolute value of the difference of the measures of the intercepted arcs.



Proof: Let $\overleftrightarrow{DC}$ be tangent to a circle at point D and let secant $\overleftrightarrow{AC}$ intersect the circle at A and B. If necessary, relabel the points so that B is between A and C. Draw $\overline{BD}$. Let $\angle 1 = \angle BCD$. Let $\angle 2 = \angle BDC$ with intercepted arc BD, and $\angle 3 = \angle ABD$ with intercepted arc AD. Since $\angle 3$ is an exterior angle of ΔBDC, $m\angle 3 = m\angle 1 + m\angle 2$ and thus, $m\angle 1 = m\angle 3 - m\angle 2$. Since $\angle 2$ is an angle formed by a tangent and a chord, $m\angle 2 = \frac{1}{2} mBD$ and since $\angle 3$ is an inscribed angle of the circle, $m\angle 3 = \frac{1}{2} mAD$. Using substitution, $m\angle 1 = \left| \frac{1}{2} mAD - \frac{1}{2} mBD \right| = \frac{1}{2}\left| mAD - mBD \right|$.

Lesson 6: Other Angles
Annotated Student Document

Review: Define each of the following terms as they relate to circles.

- Chord: A chord is a segment whose endpoints lie on a circle.

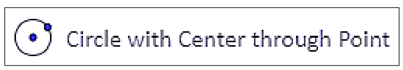
- Secant: A secant is a line that contains a chord; it intersects a circle in exactly two points.

- Tangent: A tangent is a line that intersects a circle in exactly one point.

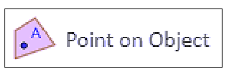
We will now explore angles that are formed by these special segments and lines.

Part 1

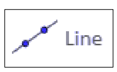
☐ Go to www.geogebra.org/classic. Be sure to turn off the grid and axes features to declutter your screen.

☐ Using the 

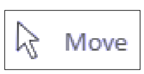
tool, create circle *c* with center *O* and point *F* on *c*. Then hide *F*.

☐ Using the 

tool, put points *A*, *B*, *C*, and *D* on *c*.

☐ Using the 

tool, connect *A* and *B* to form $\overleftrightarrow{AB}$, and connect *C* and *D* to form $\overleftrightarrow{CD}$. Notice that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are secant lines.

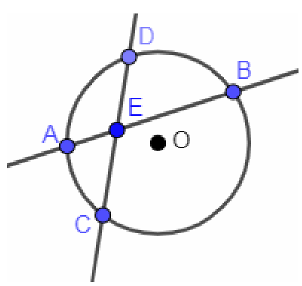
☐ Using the 

tool, move the points around *c* as necessary until the secant lines intersect inside the circle.

☐ Using the 

tool, create point *E* at the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.

Your figure should look similar to this, although your points may be elsewhere on the circle.



Consider $\angle AEC$. Since E is not necessarily the center of the circle, we cannot assume that $\angle AEC$ is a central angle of the circle. Instead, it is an angle formed by the intersection of two secants. Let's explore the measure of this angle.

☐ Using the 

tool, measure: $\angle AEC$; its intercepted arc AC; and DB, which is the intercepted arc of the vertical angle to $\angle AEC$ (namely, $\angle DEB$). Record your measurements below.

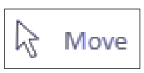
$m\angle AEC =$ _____ $\qquad$ $mAC =$ _____ $\qquad$ $mDB =$ _____

Now add the measures of the intercepted arcs together. Record the measurement below.

$mAC + mDB =$ _____

What do you observe about the relationship between $m\angle AEC$ and $mAC + mDB$?

Students should observe that the measure of the angle is half of the sum of the measures of its intercepted arcs.

☐ Use the 

tool to move points A, B, C, and D around c so that E is still inside the circle. Measure one angle formed by the intersection of the secants, as well as its intercepted arc and the intercepted arc of its vertical angle. Does your observation still hold? What does this imply about an angle formed by the intersection of two secants in the interior of a circle? Fill in the conjecture statement appropriately:

Students should note that their observation still holds.

The measure of an angle formed by two secants that intersect inside a circle is equal to half the sum of the measures of its intercepted arc and the intercepted arc of its corresponding vertical angle.

Have your teacher check your conjecture before moving on!

Part 2

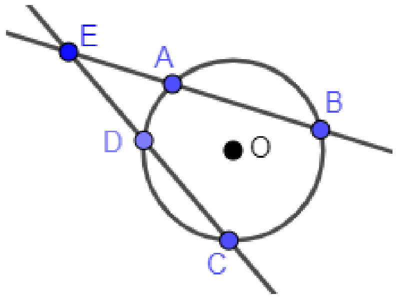
☐ In your GeoGebra worksheet, hide any angle measurements you found in Part 1.

☐ Using the figure from Part 1, use the 

tool to move points *A*, *B*, *C*, and *D* around *c* to create a new intersection point *E* that lies outside of the circle.

If necessary, relabel your points so that *A* is between *E* and *B*, and so that *D* is between *E* and *C*.

Your figure should look similar to this, although your points may be located elsewhere on the circle.



Consider $\angle CEB$. This angle is formed by the intersection of two secants outside of the circle. Let's explore the measure of this angle.

☐ Using the 

tool, measure $\angle CEB$ and its intercepted arcs AD and BC. Record your measurements below.

$m\angle CEB = \_\_\_\_\_$ $\quad mAD = \_\_\_\_\_$ $\quad mBC = \_\_\_\_\_$

In Part 1, we added the measures of the intercepted arcs together. In this part, we subtract the measure of the smaller intercepted arc from the measure of the larger intercepted arc. Subtract the measures of the intercepted arcs. Record the difference below.

$|mBC - mAD| = \_\_\_\_\_$

What do you observe about the relationship between $m\angle CEB$ and $|mBC - mAD|$?

Students should observe that the measure of the angle is half of the difference of the measures of its intercepted arcs.

- ☐ Use the Move tool to move points *A*, *B*, *C*, and *D* around *c* so that *E* is still outside of the circle. Measure the angle formed by the intersection of the secants, as well as its intercepted arcs. Does your observation still hold? What does this imply about an angle formed by the intersection of two secants outside of the circle? Fill in the conjecture statement appropriately:

  Students should note that their observation still holds.

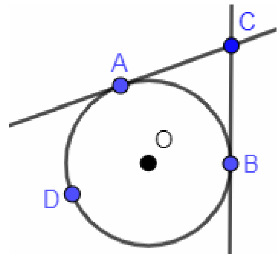
  The measure of an angle formed by two secants that intersect at a point outside a circle is equal to half the absolute value of the difference of the measures of the intercepted arcs.

STOP Have your teacher check your conjecture before moving on!

Part 3

- ☐ Open a new GeoGebra window.
- ☐ Using the Circle with Center through Point tool, create circle *c* with center *O* and point *E* on *c*. Then hide *E*.
- ☐ Using the Point on Object tool, put points *A*, *B*, and *D* on *c*.
- ☐ Using the Tangents tool, create two lines *f* and *g* that are tangent to *c* at points *A* and *B*.
- ☐ Using the Intersect tool, find the intersection of lines *f* and *g*. Label this intersection *C*.
- ☐ If necessary, move *D* so that *ADB* is a major arc.

Your figure should look similar to this, although your points of tangency may be elsewhere on the circle.



Consider $\angle ACB$. This angle is formed by the intersection of two tangents. Let's explore the measure of this angle.

☐ Using the 

tool, measure $\angle ACB$ and its intercepted arcs $\overset{\frown}{AB}$ and $\overset{\frown}{ADB}$. Record your measurements below.

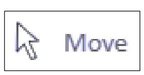
$m\angle ACB =$ _____ $\qquad m\overset{\frown}{AB} =$ _____ $\qquad m\overset{\frown}{ADB} =$ _____

Now subtract the measure of the minor intercepted arc from the measure of the major intercepted arc. Record the difference below.

$\left| m\overset{\frown}{ADB} - m\overset{\frown}{AB} \right| =$ _____

What do you observe?

Students should observe that the measure of the angle is half of the difference of the measures of its intercepted arcs.

☐ Use the 

tool to move points *A* and *B* around *c*. Measure the angle formed by the intersection of the tangents, as well as its intercepted arcs. Does your observation still hold? What does this imply about an angle formed by the intersection of two tangents? Fill in the conjecture statement appropriately:

Students should note that their observation still holds.

The measure of an angle formed by two tangents that intersect at a point outside a circle is equal to half the absolute value of the difference of the measures of the intercepted arcs.

Have your teacher check your conjecture before moving on!

Part 4

- ☐ Open a new GeoGebra window.

- ☐ Using the [Circle with Center through Point] tool, create circle *c* with center *O* and point *E* on *c*. Then hide *E*.

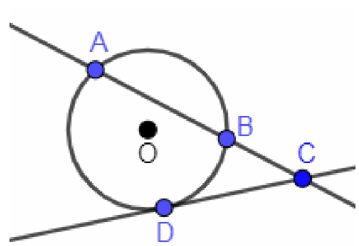
- ☐ Using the [Point on Object] tool, put points *A*, *B*, and *D* on *c*.

- ☐ Using the [Line] tool, connect *A* and *B* to form secant $\overleftrightarrow{AB}$.

- ☐ Using the [Tangents] tool, create line *g* that is tangent to *c* at *D*.

- ☐ Using the [Intersect] tool, find the intersection of the secant and tangent lines. Label this intersection *C*.

  Your figure should look similar to this, although your points may be elsewhere on the circle. If necessary, relabel points *A* and *B* such that *B* is between *A* and *C*.



  Consider $\angle BCD$. This angle is formed by the intersection of a secant and a tangent. Let's explore the measure of this angle.

- ☐ Using the [Angle] tool, measure $\angle BCD$ and its intercepted arcs AD and DB. Record your measurements below.

  $m\angle BCD =$ _____ $\qquad$ $mAD =$ _____ $\qquad$ $mDB =$ _____

  Now subtract the measures of the intercepted arcs. Record the measurement below.

  $|mAD - mDB| =$ _____

What do you observe about $m\angle BCD$ in relation to $\left|mAD - mDB\right|$?

Students should observe that the measure of the angle is half of the difference of the measures of its intercepted arcs.

☐ Use the 

tool to move points A, B, and D around c to create a new intersection point E. Measure the angle formed by the intersection of the secant and tangent, as well as its intercepted arcs. Does your observation still hold? What does this imply about an angle formed by the intersection of a secant and a tangent? Fill in the conjecture statement appropriately:

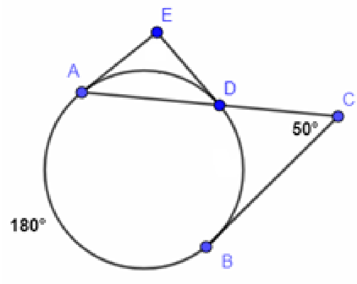
Students should note that their observation still holds.

The measure of an angle formed by a secant and a tangent is equal to half the absolute value of the difference of the measures of the intercepted arcs.

Have your teacher check your conjecture before moving on!

Use your conjectures to answer the following questions.

1. In the figure below, $\overline{EA}$, $\overline{ED}$, and $\overline{CB}$ are tangents. Find each indicated measurement.



a) mDB

Since $\angle ACB$ is an angle formed by a tangent and a secant, its measure is half the difference of its intercepted arcs so $50° = \frac{1}{2}\left(180° - mDB\right)$.

Solving this equation, we get $mDB = 80°$.

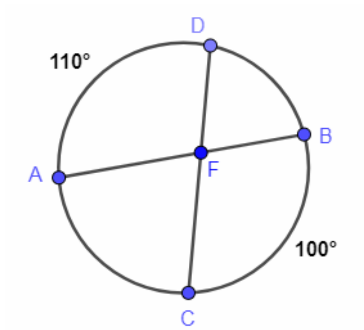
b) mAD

Since AD and DB form a semicircle,
$mAD = 180° - mDB = 180° - 80° = 100°$.

c) $m\angle AED$

Since $\angle AED$ is an angle formed by the intersection of two tangents, its measure is half the difference of its intercepted arcs so
$m\angle AED = \frac{1}{2}\left[\left(180° + 80°\right) - 100°\right] = 80°$.

2. Find $m\angle BFC$ and $m\angle CFA$.



Since $\angle BFC$ is an angle formed by the intersection of two chords, its measure is half the sum of its intercepted arcs so
$m\angle BFC = \frac{1}{2}\left(110° + 100°\right) = 105°$.

Since $\angle CFA$ and $\angle BFC$ are a linear pair,

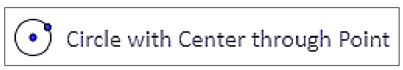
$$\begin{aligned} m\angle CFA &= 180° - m\angle BFC \\ &= 180° - 105° \\ &= 75°. \end{aligned}$$

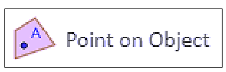
Review: Define each of the following terms as they relate to circles.

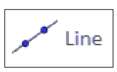
- Chord:
- Secant:
- Tangent:

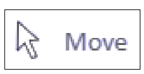
We will now explore angles that are formed by these special segments and lines.

Part 1

- ☐ Go to www.geogebra.org/classic. Be sure to turn off the grid and axes features to declutter your screen.
- ☐ Using the 

tool, create circle c with center O and point F on c. Then hide F.
- ☐ Using the 

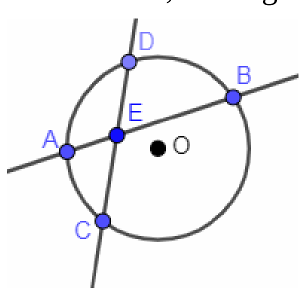
tool, put points A, B, C, and D on c.
- ☐ Using the 

tool, connect A and B to form $\overleftrightarrow{AB}$, and connect C and D to form $\overleftrightarrow{CD}$. Notice that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are secant lines.
- ☐ Using the 

tool, move the points around c as necessary until the secant lines intersect inside the circle.
- ☐ Using the 

tool, create point E at the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.

Your figure should look similar to this, although your points may be elsewhere on the circle.



Consider $\angle AEC$. Since E is not necessarily the center of the circle, we cannot assume that $\angle AEC$ is a central angle of the circle. Instead, it is an angle formed by the intersection of two secants. Let's explore the measure of this angle.

☐ Using the Angle tool, measure: $\angle AEC$; its intercepted arc AC; and DB, which is the intercepted arc of the vertical angle to $\angle AEC$ (namely, $\angle DEB$). Record your measurements below.

$m\angle AEC =$ _____ $mAC =$ _____ $mDB =$ _____

Now add the measures of the intercepted arcs together. Record the measurement below.

$mAC + mDB =$ _____

What do you observe about the relationship between $m\angle AEC$ and $mAC + mDB$?

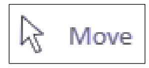
☐ Use the Move tool to move points *A*, *B*, *C*, and *D* around *c* so that *E* is still inside the circle. Measure one angle formed by the intersection of the secants, as well as its intercepted arc and the intercepted arc of its vertical angle. Does your observation still hold? What does this imply about an angle formed by the intersection of two secants in the interior of a circle? Fill in the conjecture statement appropriately:

The measure of an angle formed by two secants that intersect inside a circle is equal to ____________________ of the measures of its intercepted arc and the intercepted arc of its corresponding vertical angle.

STOP Have your teacher check your conjecture before moving on!

Part 2

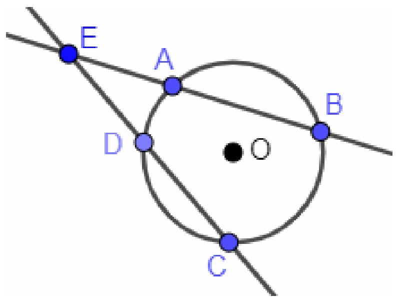
- ☐ In your GeoGebra worksheet, hide any angle measurements you found in Part 1.

- ☐ Using the figure from Part 1, use the 

tool to move points *A*, *B*, *C*, and *D* around *c* to create a new intersection point *E* that lies outside of the circle.

  If necessary, relabel your points so that *A* is between *E* and *B*, and so that *D* is between *E* and *C*.

  Your figure should look similar to this, although your points may be located elsewhere on the circle.



  Consider $\angle CEB$. This angle is formed by the intersection of two secants outside of a circle. Let's explore the measure of this angle.

- ☐ Using the 

tool, measure $\angle CEB$ and its intercepted arcs AD and BC. Record your measurements below.

  $m\angle CEB =$ _____ $\qquad$ $mAD =$ _____ $\qquad$ $mBC =$ _____

  In Part 1, we added the measures of the intercepted arcs together. In this part, we subtract the measure of the smaller intercepted arc from the measure of the larger intercepted arc. Subtract the measures of the intercepted arcs. Record the difference below.

  $\left|mBC - mAD\right| =$ _____

  What do you observe about the relationship between $m\angle CEB$ and $\left|mBC - mAD\right|$?

☐ Use the Move tool to move points *A*, *B*, *C*, and *D* around *c* so that *E* is still outside of the circle. Measure the angle formed by the intersection of the secants, as well as its intercepted arcs. Does your observation still hold? What does this imply about an angle formed by the intersection of two secants outside of the circle? Fill in the conjecture statement appropriately:

The measure of an angle formed by two secants that intersect at a point outside a circle is equal to half the absolute value of the ___________________ of the measures of the intercepted arcs.

STOP Have your teacher check your conjecture before moving on!

Part 3

☐ Open a new GeoGebra window.

☐ Using the Circle with Center through Point tool, create circle *c* with center *O* and point *E* on *c*. Then hide *E*.

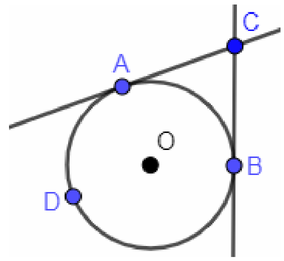
☐ Using the Point on Object tool, put points *A*, *B*, and *D* on *c*.

☐ Using the Tangents tool, create two lines *f* and *g* that are tangent to *c* at points *A* and *B*.

☐ Using the Intersect tool, find the intersection of lines *f* and *g*. Label this intersection *C*.

☐ If necessary, move *D* so that *ADB* is a major arc.

Your figure should look similar to this, although your points of tangency may be elsewhere on the circle.



Consider $\angle ACB$. This angle is formed by the intersection of two tangents. Let's explore the measure of this angle.

☐ Using the 

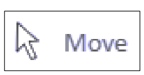
tool, measure $\angle ACB$ and its intercepted arcs $\overset{\frown}{AB}$ and $\overset{\frown}{ADB}$. Record your measurements below.

$m\angle ACB =$ _____ $\qquad m\overset{\frown}{AB} =$ _____ $\qquad m\overset{\frown}{ADB} =$ _____

Now subtract the measure of the minor intercepted arc from the measure of the major intercepted arc. Record the difference below.

$\left| m\overset{\frown}{ADB} - m\overset{\frown}{AB} \right| =$ _____

What do you observe?

☐ Use the 

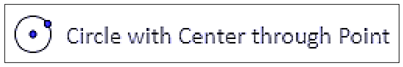
tool to move points *A* and *B* around *c*. Measure the angle formed by the intersection of the tangents, as well as its intercepted arcs. Does your observation still hold? What does this imply about an angle formed by the intersection of two tangents? Fill in the conjecture statement appropriately:

The measure of an angle formed by two tangents that intersect at a point outside a circle is equal to half the absolute value of the ___________________ of the measures of the intercepted arcs.

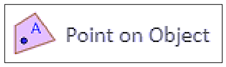
Have your teacher check your conjecture before moving on!

Part 4

- ☐ Open a new GeoGebra window.

- ☐ Using the 

tool, create circle *c* with center *O* and point *E* on *c*. Then hide *E*.

- ☐ Using the 

tool, put points *A*, *B*, and *D* on *c*.

- ☐ Using the 

tool, connect *A* and *B* to form secant $\overleftrightarrow{AB}$.

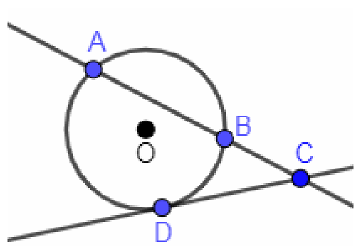
- ☐ Using the 

tool, create line *g* that is tangent to *c* at *D*.

- ☐ Using the 

tool, find the intersection of the secant and tangent lines. Label this intersection *C*.

Your figure should look similar to this, although your points may be elsewhere on the circle. If necessary, relabel points *A* and *B* such that *B* is between *A* and *C*.



Consider $\angle BCD$. This angle is formed by the intersection of a secant and a tangent. Let's explore the measure of this angle.

- ☐ Using the 

tool, measure $\angle BCD$ and its intercepted arcs AD and DB. Record your measurements below.

$m\angle BCD =$ _____ $mAD =$ _____ $mDB =$ _____

Now subtract the measures of the intercepted arcs. Record the measurement below.

$|mAD - mDB| =$ _____

What do you observe about $m\angle BCD$ in relation to $\left| mAD - mDB \right|$?

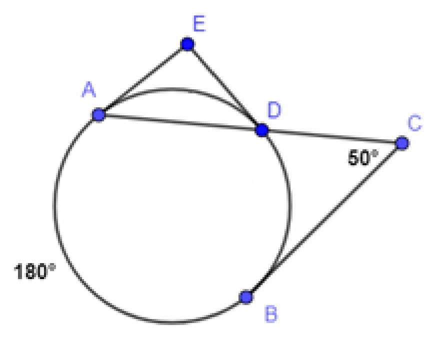
☐ Use the Move tool to move points *A*, *B*, and *D* around *c* to create a new intersection point *E*. Measure the angle formed by the intersection of the secant and tangent, as well as its intercepted arcs. Does your observation still hold? What does this imply about an angle formed by the intersection of a secant and a tangent? Fill in the conjecture statement appropriately:

The measure of an angle formed by a secant and a tangent is equal to half the absolute value of the _______________ of the measures of the intercepted arcs.

Have your teacher check your conjecture before moving on!

Use your conjectures to answer the following questions.

1. In the figure below, $\overline{EA}$, $\overline{ED}$, and $\overline{CB}$ are tangents. Find each indicated measurement.

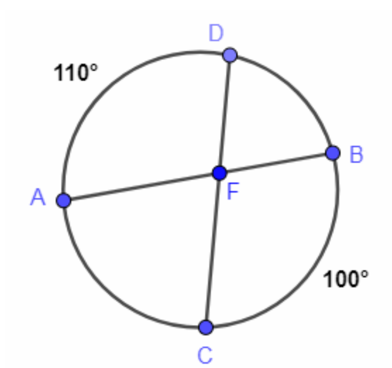


a) mDB

b) mAD

c) $m\angle AED$

2. Find $m\angle BFC$ and $m\angle CFA$.



Lesson 7: Circles and Lengths of Segments
Teacher Overview

Introduction: This lesson helps students to discover relationships between lengths of special segments in a circle. Students will work in pairs, using GeoGebra. The following theorems are covered.

- When two chords intersect, the product of the lengths of the segments of one chord equals the product of the lengths of the segments of the other chord.
- When two secant segments are drawn to a circle from an external point, the product of one secant segment's length and its external segment's length equals the product of the other secant segment's length and its external segment's length.
- When a secant and a tangent are drawn to a circle from an external point, the product of the secant segment's length and its external segment's length is equal to the square of the tangent segment's length.

Pre-requisite Knowledge:
- The measure of an angle that is inscribed in a circle is equal to half the measure of the angle's intercepted arc.
- The measure of an angle that is formed by a tangent and a chord is equal to half the measure of the angle's intercepted arc.
- Angle-Angle Similarity
- Inscribed angles that intercept the same arc are congruent.

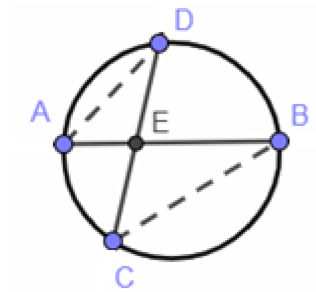
Allotted Time: 1 class period (40 minutes)

Materials:
- For Students
  - Student Handout (1 per student)
  - Technology device with internet capability (at least 1 per pair of students)
- For Teacher
  - Teacher Notes
  - Annotated Student Handout

Lesson Overview:
- Students work in pairs on Student Handout. (20 minutes)
  - Monitor student work by circulating around the classroom. If students are struggling, ask pointed questions to get them back on track. Refrain from giving students direct answers, as this is meant to be a discovery activity.
  - As students form their conjectures, check to make sure the conjectures are accurate. If not, redirect students in a helpful manner to allow them to find the correct conclusions.
  - Once their conjectures have been checked, students can work on the sample problems to practice the concepts they learned.

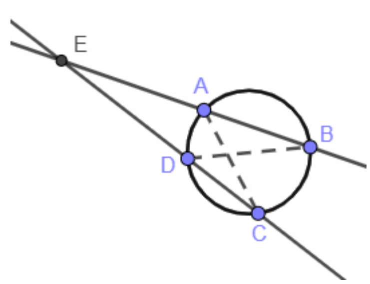
- Bring students back into a whole group setting and go over the proofs of these conjectures. (15 minutes)
    - Use the proofs on the following page to lead the discussion, but ask for student input along the way.
    - Students should follow along and take notes on the proofs if necessary.
- Conclude the lesson by reviewing the sample problems and assigning homework problems related to the lesson content. (5 minutes)

Theorem: When two chords intersect, the product of the lengths of the segments of one chord equals the product of the lengths of the segments of the other chord.



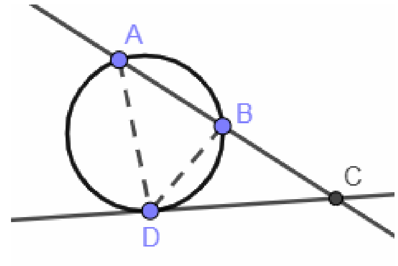
Proof: Let $\overline{AB}$ and $\overline{CD}$ be chords of a circle that intersect at point E inside the circle. Draw $\overline{AD}$ and $\overline{CB}$. Notice that $\angle DAB \cong \angle DCB$ and $\angle ABC \cong \angle ADC$ because inscribed angles that intercept the same arc are congruent. Now, $\Delta EAD \sim \Delta ECB$ by the Angle-Angle Similarity Postulate. By the definition of similarity, corresponding sides are in proportion so $\frac{EA}{EC} = \frac{ED}{EB}$ and multiplying both sides of the equation by $EC \cdot EB$ gives $EA \cdot EB = EC \cdot ED$.

Theorem: When two secant segments are drawn to a circle from an external point, the product of one secant segment's length and its external segment's length equals the product of the other secant segment's length and its external segment's length.



Proof: Let A, B, C and D be points on a circle such that secants $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$ intersect at point E outside of the circle. Relabel the points if necessary so that A is between B and E, and D is between C and E. Draw $\overline{AC}$ and $\overline{DB}$. Notice $\angle BAC \cong \angle BDC$ because inscribed angles that intercept the same arc are congruent. Also, $\angle AEC \cong \angle BED$ so $\Delta EAC \sim \Delta EDB$ by the Angle-Angle Similarity Postulate. By the definition of similarity, corresponding sides are in proportion so $\frac{EC}{EB} = \frac{EA}{ED}$ and multiplying both sides of the equation by $EB \cdot ED$ gives $EC \cdot ED = EB \cdot EA$.

Theorem: When a secant and a tangent are drawn to a circle from an external point, the product of the secant segment's length and its external segment's length equals the square of the tangent segment's length.



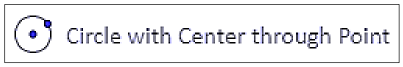
Proof: Let $\overleftrightarrow{AB}$ be a secant to a circle that contains chord $\overline{AB}$. Let $\overleftrightarrow{DC}$ be a tangent to the circle at *D*, which intersects $\overleftrightarrow{AB}$ at point *C* outside of the circle. Relabel the points if necessary so that *B* is between *A* and *C*. Draw $\overline{DA}$ and $\overline{DB}$. Notice that $\angle BAD$ is an inscribed angle, so $m\angle BAD$ is half the measure of the intercepted arc $\overset{\frown}{BD}$. Also, $\angle BDC$ is formed by a chord and tangent, so $m\angle BDC = \frac{1}{2} m\overset{\frown}{BD}$ by a theorem in Lesson 5. Then $\angle BAD \cong \angle BDC$, so $\Delta CAD \sim \Delta CDB$ by the Angle-Angle Similarity Postulate. By the definition of similarity, corresponding sides are in proportion so $\frac{AC}{DC} = \frac{DC}{BC}$ and multiplying both sides of the equation by $DC \cdot BC$ gives $AC \cdot BC = (DC)^2$.

Lesson 7: Circles and Lengths of Segments
Annotated Student Document

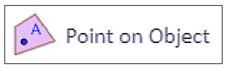
In the previous lesson, we explored special angles created by the intersection of chords, tangents, and secants. We will now explore lengths of segments under these conditions.

Part 1

- ☐ Go to www.geogebra.org/classic. Be sure to turn off the grid and axes features to declutter your screen.

- ☐ Using the 

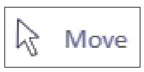
tool, create circle c with center O and point F on c. Then hide O and F.

- ☐ Using the 

tool, put points A, B, C, and D on c.

- ☐ Using the 

tool, connect A and B to form $\overleftrightarrow{AB}$, and connect C and D to form $\overleftrightarrow{CD}$. Notice that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are secant lines.

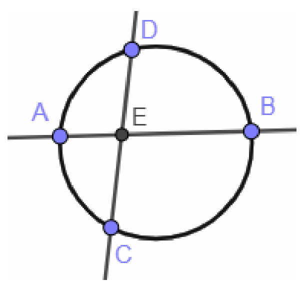
- ☐ Using the 

tool, move the points around c as necessary until the secant lines intersect inside the circle.

- ☐ Using the 

tool, create point E at the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.

Your figure should look similar to this, although your points may be elsewhere on the circle.



Consider the two chords, $\overline{AB}$ and $\overline{CD}$. Notice that each chord gets split into two segments by the intersection point E. These segments are called **the segments of the chords**. Based on this information, fill in the blanks below.

The segments of $\overline{AB}$ are $\overline{EA}$ and $\overline{EB}$.

The segments of $\overline{CD}$ are $\overline{EC}$ and $\overline{ED}$.

- ☐ Using the [Distance or Length] tool, measure each segment of each chord. Record your measurements below.

  $EA =$ _____ $EB =$ _____

  $EC =$ _____ $ED =$ _____

  Now multiply the lengths of the segments of each chord. Record the measurements below.

  $EA \cdot EB =$ _____

  $EC \cdot ED =$ _____

  What do you observe?

  Students should observe that the products are equal.

- ☐ Use the [Move] tool to move points *A*, *B*, *C*, and *D* around *c* to create a new intersection point *E* inside the circle. Measure the segments of the chords and find the corresponding products. Does your observation still hold? What does this imply about the products of the segments of two intersecting chords? Fill in the conjecture statement appropriately:

  Students should note that their observation still holds.

  When two chords intersect, the product of the lengths of the segments of one chord equals the product of the lengths of the segments of the other chord.

STOP Have your teacher check your conjecture before moving on!

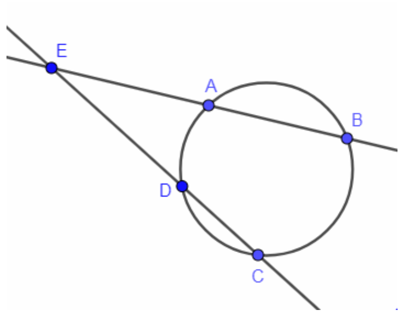
Part 2

- ☐ In your GeoGebra worksheet, hide any measurements you found in Part 1.

- ☐ Using the figure from Part 1, use the [Move] tool to move points *A*, *B*, *C*, and *D* around *c* to create a new intersection point *E* that lies outside of the circle.

  If necessary, relabel your points so that *A* is between *E* and *B*, and so that *D* is between *E* and *C*.

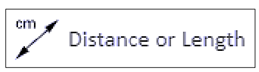
Your figure should look similar to this, although your points may be located elsewhere on the circle.



The segments $\overline{EB}$ and $\overline{EC}$ are part of secants $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$ so we call them **secant segments**. Notice that each secant segment is split into two parts, one segment outside of the circle and one segment inside the circle. For our discussion, we will be concerned with the part of the secant segment outside of the circle, known as the **external segment of the secant segment.** Based on this information, fill in the blanks below.

The external segment of secant segment $\overline{EB}$ is $\overline{EA}$.

The external segment of secant segment $\overline{EC}$ is $\overline{ED}$.

☐ Using the 

tool, measure each secant segment and its external segment. Record the measurements below.

$EB =$ ______ $EA =$ ______

$EC =$ ______ $ED =$ ______

Now multiply the measure of each secant segment by the measure of its external segment. Record the measurements below.

$EB \cdot EA =$ ______

$EC \cdot ED =$ ______

What do you observe?

Students should observe that the products are equal.

☐ Use the Move tool to move points *A*, *B*, *C*, and *D* around *c* to create a new intersection point *E* outside the circle. Measure the secant segments and their external segments, then compute the products again. Does your observation still hold? What does this imply about the products of two secant segments and their external segments? Fill in the conjecture statement appropriately:

Students should note that their observation still holds.

When two secant segments are drawn to a circle from an external point, the product of one secant segment's length and its external segment's length equals the product of the other secant segment's length and its external segment's length.

STOP Have your teacher check your conjecture before moving on!

Part 3

☐ Open a new GeoGebra window.

☐ Using the Circle with Center through Point tool, create circle *c* with center *O* and point *E* on *c*. Then hide *O* and *E*.

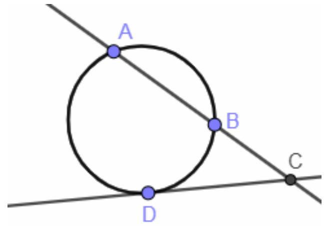
☐ Using the Point on Object tool, put points *A*, *B*, and *D* on *c*.

☐ Using the Line tool, connect *A* and *B* to form secant $\overleftrightarrow{AB}$.

☐ Using the Tangents tool, create line *g* that is tangent to circle *c* at *D*.

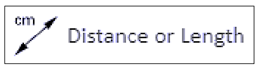
☐ Using the Intersect tool, find the intersection of the secant and tangent lines. Label this intersection *C*.

Your figure should look similar to this, although your points may be elsewhere on the circle. If necessary, relabel points *A* and *B* such that *B* is between *A* and *C*.



What is the external segment of secant segment $\overline{AC}$? $\overline{BC}$

Notice that segment $\overline{DC}$ is part of tangent $\overleftrightarrow{DC}$. We call this a **tangent segment**.

☐ Using the 

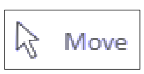
tool, measure the secant segment and its external segment, and measure the tangent segment. Record the measurements below.

$AC =$ _____ $BC =$ _____ $DC =$ _____

Now multiply the measure of the secant segment by the measure of its external segment, and square the measure of the tangent segment. Record the measurements below. What do you observe?

$EB \cdot EA =$ _____ $(EF)^2 =$ _____

Students should observe that the products are equal.

☐ Use the 

tool to move points *A*, *B*, and *D* around *c* to create a new intersection point *C*. Measure the secant segment and its external segment, and measure the tangent segment. Then compute the products again. Does your observation still hold? What does this imply about the product of a secant segment and its external segment compared to the square of a tangent segment? Fill in the conjecture statement appropriately:

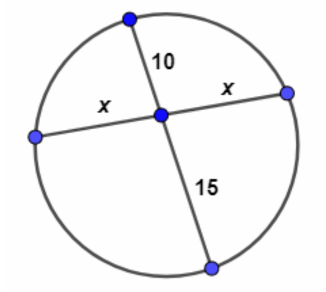
Students should note that their observation still holds.

When a secant and a tangent are drawn to a circle from an external point, the product of the secant segment's length and its external segment's length equals the square of the tangent segment's length.

Have your teacher check your conjecture before moving on!

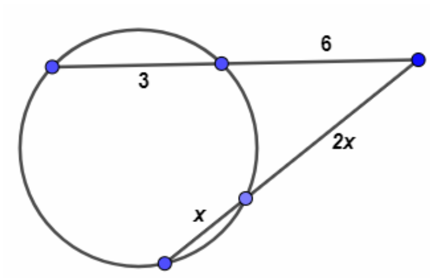
Use your conjectures to find the value of *x* in each of the following questions.

1.



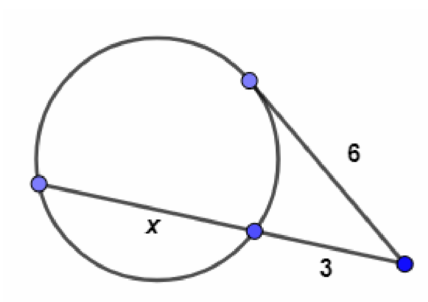
The products of the segments of each chord are equal, so $x \cdot x = 10 \cdot 15$. Thus, $x = \sqrt{150} = 5\sqrt{6}$.

2.



The product of one secant segment and its external segment is equal to the product of the other secant segment and its external segment, so $2x(2x+x) = 6(3+6)$. Thus, $x = 3$.

3.



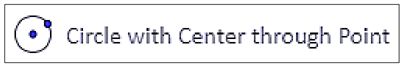
The product of the secant segment and its external segment is equal to the square of the tangent segment, so $3(x+3) = 6^2$. Thus, $x = 9$.

Lesson 7: Circles and Lengths of Segments
Student Document

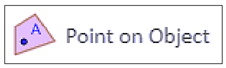
In the previous lesson, we explored special angles created by the intersection of chords, tangents, and secants. We will now explore lengths of segments under these conditions.

Part 1

- ☐ Go to www.geogebra.org/classic. Be sure to turn off the grid and axes features to declutter your screen.

- ☐ Using the 

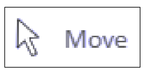
tool, create circle *c* with center *O* and point *F* on *c*. Then hide *O* and *F*.

- ☐ Using the 

tool, put points *A*, *B*, *C*, and *D* on *c*.

- ☐ Using the 

tool, connect *A* and *B* to form $\overleftrightarrow{AB}$, and connect *C* and *D* to form $\overleftrightarrow{CD}$. Notice that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are secant lines.

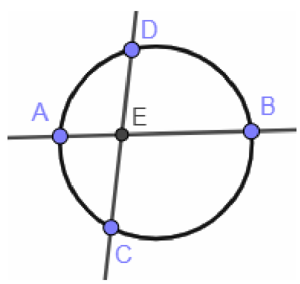
- ☐ Using the 

tool, move the points around *c* as necessary until the secant lines intersect inside the circle.

- ☐ Using the 

tool, create point *E* at the intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.

Your figure should look similar to this, although your points may be elsewhere on the circle.



Consider the two chords, $\overline{AB}$ and $\overline{CD}$. Notice that each chord gets split into two segments by the intersection point *E*. These segments are called **the segments of the chords**. Based on this information, fill in the blanks below.

The segments of $\overline{AB}$ are _____ and _____.

The segments of $\overline{CD}$ are _____ and _____.

☐ Using the Distance or Length tool, measure each segment of each chord. Record your measurements below.

$EA =$ ______ $EB =$ ______

$EC =$ ______ $ED =$ ______

Now multiply the lengths of the segments of each chord. Record the measurements below.

$EA \cdot EB =$ ______

$EC \cdot ED =$ ______

What do you observe?

☐ Use the Move tool to move points *A*, *B*, *C*, and *D* around *c* to create a new intersection point *E* inside the circle. Measure the segments of the chords and find the corresponding products. Does your observation still hold? What does this imply about the products of the segments of two intersecting chords? Fill in the conjecture statement appropriately:

When two chords intersect, the product of the lengths of the segments of one chord ________ the product of the lengths of the segments of the other chord.

STOP Have your teacher check your conjecture before moving on!

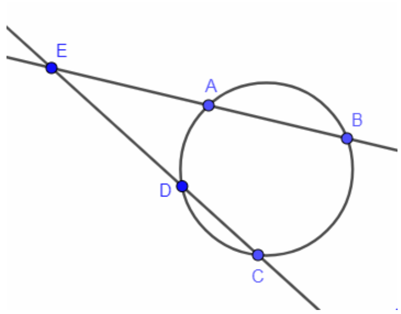
Part 2

☐ In your GeoGebra worksheet, hide any measurements you found in Part 1.

☐ Using the figure from Part 1, use the Move tool to move points *A*, *B*, *C*, and *D* around *c* to create a new intersection point *E* that lies outside of the circle.

If necessary, relabel your points so that *A* is between *E* and *B*, and so that *D* is between *E* and *C*.

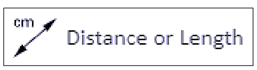
Your figure should look similar to this, although your points may be located elsewhere on the circle.



The segments $\overline{EB}$ and $\overline{EC}$ are part of secants $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$ so we call them **secant segments**. Notice that each secant segment is split into two parts, one segment outside of the circle and one segment inside the circle. For our discussion, we will be concerned with the part of the secant segment outside of the circle, known as the **external segment of the secant segment.** Based on this information, fill in the blanks below.

The external segment of secant segment $\overline{EB}$ is _____.

The external segment of secant segment $\overline{EC}$ is _____.

☐ Using the 

tool, measure each secant segment and its external segment. Record the measurements below.

$EB =$ ______ $EA =$ ______

$EC =$ ______ $ED =$ ______

Now multiply the measure of each secant segment by the measure of its external segment. Record the measurements below.

$EB \cdot EA =$ ______

$EC \cdot ED =$ ______

What do you observe?

- ☐ Use the Move tool to move points *A*, *B*, *C*, and *D* around *c* to create a new intersection point *E* outside the circle. Measure the secant segments and their external segments, then compute the products again. Does your observation still hold? What does this imply about the products of two secant segments and their external segments? Fill in the conjecture statement appropriately:

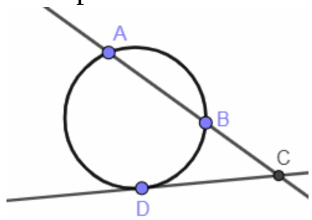
When two secant segments are drawn to a circle from an external point, the product of one secant segment's length and its external segment's length __________ the product of the other secant segment's length and its external segment's length.

STOP Have your teacher check your conjecture before moving on!

Part 3

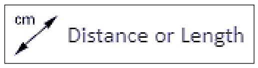
- ☐ Open a new GeoGebra window.
- ☐ Using the Circle with Center through Point tool, create circle *c* with center *O* and point *E* on *c*. Then hide *O* and *E*.
- ☐ Using the Point on Object tool, put points *A*, *B*, and *D* on *c*.
- ☐ Using the Line tool, connect *A* and *B* to form secant $\overleftrightarrow{AB}$.
- ☐ Using the Tangents tool, create line *g* that is tangent to circle *c* at *D*.
- ☐ Using the Intersect tool, find the intersection of the secant and tangent lines. Label this intersection *C*.

Your figure should look similar to this, although your points may be elsewhere on the circle. If necessary, relabel points *A* and *B* such that *B* is between *A* and *C*.



What is the external segment of secant segment $\overline{AC}$? _____

Notice that segment $\overline{DC}$ is part of tangent $\overleftrightarrow{DC}$. We call this a **tangent segment**.

☐ Using the 

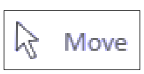
tool, measure the secant segment and its external segment, and measure the tangent segment. Record the measurements below.

$AC =$ _____ $BC =$ _____ $DC =$ _____

Now multiply the measure of the secant segment by the measure of its external segment, and square the measure of the tangent segment. Record the measurements below.

$EB \cdot EA =$ _____ $(EF)^2 =$ _____

What do you observe?

☐ Use the 

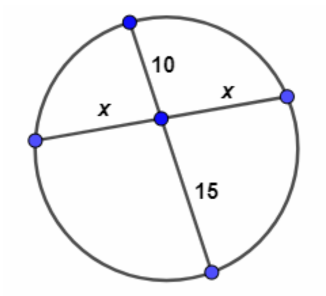
tool to move points *A*, *B*, and *D* around *c* to create a new intersection point *C*. Measure the secant segment and its external segment, and measure the tangent segment. Then compute the products again. Does your observation still hold? What does this imply about the product of a secant segment and its external segment compared to the square of a tangent segment? Fill in the conjecture statement appropriately:

When a secant and a tangent are drawn to a circle from an external point, the product of the secant segment's length and its external segment's length __________ the square of the tangent segment's length.

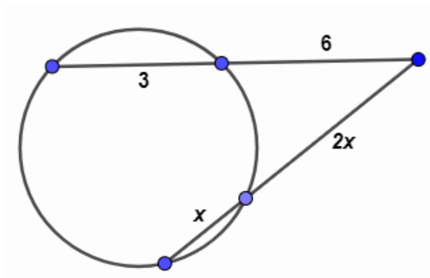
Have your teacher check your conjecture before moving on!

Use your conjectures to find the value of *x* in each of the following questions.

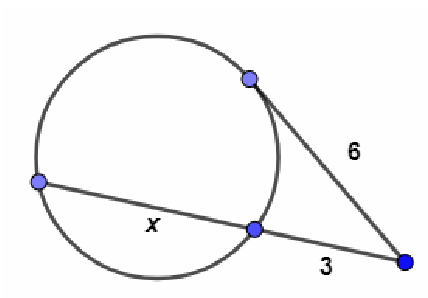
1.



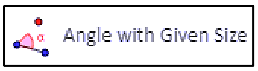
2.



3.

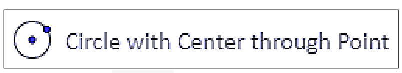


: The angle tool is located under the  dropdown menu. To utilize the angle tool, select three points or two lines to find the measure of the angle created. **Note:** The angle measure is that of the angle formed in a counterclockwise direction, so order of points or lines selected matters. If you measure in the wrong direction, just undo and measure again.

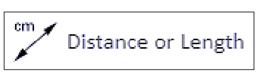


: The angle with given size tool is located under the  dropdown menu. To utilize the angle with given size tool, select a leg point and a vertex point for the angle. Then type in the desired angle measure and choose whether the angle should be counterclockwise or clockwise around the vertex. The angle measurement can be a number or a variable.

: The circle with center and radius tool is located under the  dropdown menu. To utilize the circle with center and radius tool, select a center point for the circle and then type in a radius measure. The radius can be a number or a variable.

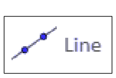


: The circle with center through point tool is located under the  dropdown menu. To utilize the circle with center through point tool, select a center point for the circle and then a point on the circle.

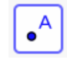


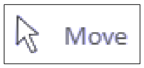
: The distance or length tool is located under the  dropdown menu and measures the distance between two points, or the length of a segment. To utilize the distance or length tool, select two points or a segment.

: The intersect tool is located under the  dropdown menu. To utilize the intersect tool, select objects successively to create an intersection point. If you click near enough to an apparent intersection point, you will select both objects simultaneously.

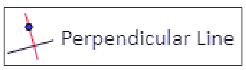


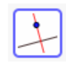
: The line tool is located under the  dropdown menu. To utilize the line tool, select two points or two positions to connect with a line.

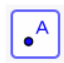
: The midpoint or center tool is located under the  dropdown menu and constructs the midpoint of a segment (regardless of whether the segment has been constructed), or the center of a circle or polygon. To utilize the midpoint or center tool, select two points, or a constructed segment, circle, or polygon.

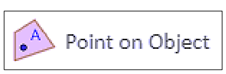


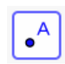
: The move tool is located under the  dropdown menu. To utilize the move tool, select an object and drag it around the screen to move it to any location. **Note:** Objects which rely on other objects (e.g. intersection points) cannot be moved with the move tool.



: The perpendicular line tool is located under the  dropdown menu. To utilize the perpendicular line tool, select a line and a point from which to draw the perpendicular.

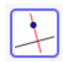
: The point tool is located under the  dropdown menu. To utilize the point tool, select a position and click. **Note:** The point tool is convenient to use when the point does not need to be fixed to anything.



: The point on object tool is located under the  dropdown menu. To utilize the point on object tool, select an object or its perimeter and click where the point should go. **Note:** The point on object tool is more specific than the point tool. It should be used when specifically fixing a point onto an object.

: The segment tool is located under the  dropdown menu. To utilize the segment tool, select two pre-existing points to connect or click on two positions to create points and a line segment connecting them.

: The segment with given length tool is located under the  dropdown menu. To utilize the segment with given length tool, select an endpoint for the segment and then type in the desired length of the segment. **Note:** The segment automatically appears horizontal. To adjust the segment's position, use the move tool to move the endpoint that was created.

: The tangents tool is located under the  dropdown menu. To utilize the tangents tool, select the circle and then a point through which the tangent will go.