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ANCIENT CULTURES + HIGH SCHOOL ALGEBRA = A DIVERSE MATHEMATICAL APPROACH

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The essay of Laryssa J. Byndas is hereby accepted:

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0. INTRODUCTION

High school mathematics teachers are always looking for ways to excite and intrigue students with material that is new or different. Many people have the perception that mathematics is rigid and have the opinion that there is just one way to attack a particular problem and there is just one correct solution. The tendency is to forget, or not even realize, that our modern mathematical techniques have developed over time. We must give credit to the ancient mathematicians who used different notations, languages, media, and procedures for how we learn, solve, and think about math in today's world. The main goal of this essay is to look at some topics in high school Algebra from the perspective of ancient cultures. Are the ideas completely different from what is known today? No, because much of what is taught and learned today is based on work that was done long ago. But do these methods provide a different perspective? Absolutely.

The reader who hopes to gain knowledge of the history of mathematicians from these ancient cultures and the political and social ideals they lived by can find that information at a public or college library. Similarly, if the reader's goal is to explore very basic counting or higher level mathematics, they should look elsewhere. But if the reader's goal is to find ways of incorporating mathematics from these different cultures into the high school Algebra curriculum, then this paper is a good start. It is easy to find plenty of books on the history of mathematics and there are books that walk the reader through some very low level as well as high level math. It is much harder to find a single source of information like that in this paper: topics that originate in ancient cultures and can be easily understood and realistically used in a high school class. Mathematics instructors can choose a topic from this paper, read a quick background of the lesson, and examine the teacher guide and annotated student worksheets once or twice to completely understand the procedure. They will then have a lesson that efficiently and seamlessly places the topic into their own classes.

Students are all unique and come from different backgrounds. It is important for students to know that what and how they learn in school is not always what was done in the past. It is also important for students to know that there is always more than one way to look at mathematics. The topics in this paper may challenge students with new thought processes. Students may finally be able to solve a problem that they could not before, be able to let go of that calculator crutch that our culture allows them to use, or become interested in topics by use of technology. Regardless, students can discover something new and learn the lesson that there is always another way. Always.

1. METHOD OF FALSE POSITION: EGYPTIAN

Background and Teacher Notes

In this lesson, students learn about the Method of False Position, which is a technique for solving linear equations of the form $ax = b$. This technique is found in the Ahmes Papyrus, also known as the Rhind Mathematical Papyrus, which is the major source of current knowledge about Egyptian mathematics. The Ahmes Papyrus was written in about 1650 BC and was based on a work that was composed 300 years previously [5, p. 60]. Mathematics in ancient Egypt is difficult to research because of the lack of original sources that have been found on Egyptian mathematics. Translating these is difficult, considering that the sources were written in hieratic. This is a symbolic notational system, which is not commonly read and understood in modern times, that was used to create the Ahmes Papyrus [6, p. 61].

Some say that Geometry actually originated in Egypt due to construction of the pyramids and the need to resurvey land after flooding, but the beginnings of algebra can also be found in these early writings. Generally, the Egyptians verbalized all mathematical rules because there was no standard mathematical notation. Egyptian mathematical documents simply used lists of instructions to describe solutions to problems instead of creating formulas, though one can see that this is an early form of what we do today.

Solving linear equations is a process that students repeat throughout most of the high school curriculum. Especially in Algebra 1, students struggle with the idea of getting the variable by itself and solving in the traditional modern manner. The Method of False Position gives students an alternative method, as well as challenges them to think outside of the box. It eliminates the robotic process that so many students use to attack a linear equation. For this lesson, we assume that students have learned the traditional modern method of solving a linear equation. This lesson is meant to be led by the teacher at the beginning to gain the interest of the students and explain the process. Students will then work with each other to test this process and decide whether it always works.

Teacher Guide

The Egyptians knew how to solve what are now known as linear equations. Specifically they had a method for solving equations of the form $x + ax = b$ or $x + ax + bx = c$, where *a*, *b*, and *c* are given constants and *x* is the unknown. Most commonly, the Egyptian method of solving these problems is now called the "Method of False Position." This method can be broken down into these steps:

1. Guess a value for the unknown. After demonstrating the process, make sure students realize why the guess should not be equal to zero. (If the guess is $x = 0$, any multiple of *x* will result in zero, meaning we cannot find or use a proportionality factor as described below).
2. Complete the steps described in the problem, using this guess.
3. Compare the result of using the guess with the desired result and determine the proportionality factor (i.e., what must we multiply the assumed result by in order to get the desired result?).
4. Multiply the original guess by the proportionality factor to determine the correct value of the unknown.
5. Check the solution.

<u>Example from the Lesson Plan</u>

- Ask a student to guess a solution to $5x + \frac{1}{2}x = 22$ that is not 4; say the student chooses 2.
- Investigate what happens if we let $x = 2$ on the left side of the equation

$$5(2) + \frac{1}{2}(2) = 10 + 1 = 11 .$$

- Notice that the correct solution cannot be 2, because $11 \neq 22$.

- Ask students to find the ratio of 22 to 11. Students should answer $\frac{22}{11} = 2$.
- Multiply the original guess by this ratio. We get $2\left(\frac{22}{11}\right) = 4$, which is the correct solution!

This process is an excellent way to provide students with an alternate method of solving linear equations. It also is a nice application of using proportional reasoning.

<u>Proof of the Method of False Position</u>

To use this method, the equation must have two equal expressions; one that simplifies to the form *ax* and the other that simplifies to a constant. That is, the simplified form of the equation must be $ax = b$.

Consider the equation $ax = b$. Let $g \neq 0$ be the initial guess. Then the proportion of b to ag is $\frac{b}{ag}$, so when we multiply the guess by this proportion, we get $g\left(\frac{b}{ag}\right) = \frac{b}{a}$. This is indeed the solution to $ax = b$.

<u>Additional Examples</u>

For teachers who want to create more problems that use the Method of False Position, we suggest the following:

- Any equation must simplify to $ax = b$. See Question 1.d) in the student worksheet.
- Use some equations whose solutions are not whole numbers. This reduces the chance of students' guessing the correct solution instead of using the Method of False Position.

Lesson Plan

Goals

- Solve a linear equation using a new method.
- Examine and discover why this method works.

Introduction

Problem: Solve the following linear equation for *x*: $5x+\frac{1}{2}x=22$.

Give students time to solve; the solution is $x=4$.

Students will typically solve this problem using fractions or decimals.

Discuss these questions with the class.

- Why did students do the problem the way they did?
- Is it just because we know the process?
- Is there another way we could have solved this?
- Do we like using decimals and fractions?

Ask students whether there is a way to solve this equation without using algebra.

New Material

Begin with the example provided in the Teacher Guide.

At this point, students should be intrigued and will probably be wondering if it would work if they picked another random guess. Have each group choose a different number as their initial guess and try the process. Have them share which number they chose and ask them if the process worked.

Hopefully at some point, a group will guess an odd number. That should lead to a good discussion of why we could start with an odd number, but it is easier to use even numbers since our guess will be multiplied by $\frac{1}{2}$. If no student starts with an odd number, the teacher should invite a discussion.

Before students begin their worksheet, use the information about the Rhind Mathematical Papyrus in the Background and Teacher Notes section above to give the students a little information about what it is, where it is from, and how it was written.

Annotated Student Worksheet

1. Solve the following problems using the Method of False Position. Show all work!

a) Problem 25 from the Ahmes Papyrus [3]
A quantity and its $\frac{1}{2}$ added together become 16. What is the quantity?

Initial Guess from the Ahmes Papyrus: 2

$$2+\frac{1}{2}(2)=2+1=3$$

Correct Answer: $2\left(\frac{16}{3}\right)=\frac{32}{3}$

Check Answer: $\frac{32}{3}+\frac{1}{2}\left(\frac{32}{3}\right)=\frac{32}{3}+\frac{16}{3}=\frac{48}{3}=16$

b) Problem 27 from the Ahmes Papyrus [3]
A quantity and its $\frac{1}{5}$ added together become 21. What is the quantity?

Initial Guess from the Ahmes Papyrus: 5

$$5+\frac{1}{5}(5)=5+1=6$$

Correct Answer: $5\left(\frac{21}{6}\right)=\frac{35}{2}$

c) Solve for *x*: $\frac{1}{3}x+\frac{1}{5}x+x-\frac{1}{2}\left(x+\frac{1}{3}x\right)=5$

Initial Guess: 30

$$\frac{1}{3}(30)+\frac{1}{5}(30)+30-\frac{1}{2}\left(30+\frac{1}{3}(30)\right)=26$$

Correct Answer: $30\left(\frac{5}{26}\right)=\frac{75}{13}$

d) Solve for *x*: $x+\frac{2}{3}x+10=110$ *

Initial Guess: 3

$$3+\frac{2}{3}(3)+10=3+2+10=15$$

"Correct Answer:" $3\left(\frac{110}{15}\right)=22$

Check Answer: $22+\frac{2}{3}(22)+10=\frac{140}{3}$ **NOTICE:** $\frac{140}{3}\neq 110$

*This equation cannot be solved directly using the Method of False Position since it cannot be simplified to the form $ax=b$. Discuss this as a class. A quick proof of the validity of the Method of False Position is located for teacher use in the teacher guide.

<u>Instead:</u> First simplify: $x+\frac{2}{3}x=100$

Initial Guess: 3

$$3+\frac{2}{3}(3)=3+2=5$$

$$3\left(\frac{100}{5}\right)=60$$

Check Answer: $60+\frac{2}{3}(60)+10=60+40+10=110$

e) Two-sevenths of a quantity is subtracted from its double and together become 80. What is the quantity?

Initial Guess: 70

The double of 70 is 140, and $\frac{2}{7}$ of 70 is 20. So we get $140 - 20 = 120$.

Correct Answer: $70\left(\frac{80}{120}\right) = \frac{140}{3}$.

2. Give some reasons why, or situations when, the Method of False Position would be more efficient or better than modern algebraic methods.

Potential Answers: The work can be more efficient with fewer steps.
We do not have to use decimals or fractions.
We do not have to use a calculator.

References and Further Reading

[1] Boyer, Carl B., and Merzback, Uta C. *A History of Mathematics*. John Wiley & Sons, Inc., 2011: pp. 8-20.

[3] Chace, Arnold B. *The Rhind Mathematical Papyrus Volume 1: Free Translation and Commentary*. Mathematical Association of America, 1927. <https://upload.wikimedia.org/wikipedia/commons/7/7b/The_Rhind_Mathematical_Papyrus,_Volume_I.pdf>: pp. 68-69.

[5] Joseph, George G. *The Crest of the Peacock: Non-European Roots of Mathematics*. Princeton University Press, 2000: pp. 57-90.

2. DIVISION BY ZERO: INDIAN

Background and Teacher Notes

There is proof of ancient civilizations in India that show evidence of a cultured civilization dating back to 2650 BCE, but there are no recovered Indian mathematical documents from that time period. One of the probable reasons for this is due to the written languages and dialects that developed from conquests over parts of India that have yet to be deciphered [1, p. 186]. Some proof of mathematical thinking can be found in ancient Indian texts called the *Vedas*. Even though these are mostly religious texts, they contain some mathematical topics as well.

Indian mathematicians were one of the first groups of people to use, understand, and investigate the number zero. The Indian mathematician Bhaskara II, who lived in the 12^{th} century, is credited with being able to explain why there is no division by zero. In this lesson, students explore why dividing by zero is undefined, rather than relying on the fact that they have always been told it is impossible or because the calculator produced an error. They can create a physical understanding of why this actually is!

Though division is not a standard in the high school curriculum, this lesson involves proofs, which are mathematically placed throughout the high school curriculum, and introduces the idea of limits and sequences, which are found in higher level high school mathematics. This lesson also serves as a good refresher on the meaning of division as well as strengthening students' understanding of division by fractions. This lesson, which can be used as an extension activity, gets students to think deeply and prove a mathematical idea. It teaches them not to just accept a fact they have been told over and over again. For differentiation purposes, students can produce a formal proof or develop less formal reasoning as to why division by zero is impossible.

Teacher Guide

In this lesson, students should explore the sequence $5 \div \frac{1}{10}, 5 \div \frac{1}{100}, 5 \div \frac{1}{1000}, \ldots,$ so begin this lesson with a discussion of division. Not only is this beneficial to students for a refresher but students must understand division by fractions in order to be able to follow the mathematics involved. Students will discover that as the denominator in the divisor increases, the divisor gets closer and closer to zero while the quotients get arbitrarily large. They will explore this idea by thinking about the limit of the sequence.

According to Bhaskara II, we would say that $5 \div 0 = \infty$. Even though this statement makes sense based upon his work, we know today that division by zero does not result in infinity. Students will report that $5 \div 0$ is undefined, but do they understand why it is not "infinity"? This is an excellent discussion topic for students, dependent on the students' level. The reason why we teach students that the result of any number divided by zero is undefined (and not infinity) is due to the fact that Bhaskara II's proof is not complete.

Consider using a negative number as the divisor such as $5 \div \left(\frac{1}{-10}\right)$, $5 \div \left(\frac{1}{-100}\right)$, $5 \div \left(\frac{1}{-1000}\right)$, … . In this situation we determine that the sequence tends toward negative infinity. In a Pre-Calculus class, this is a wonderful use of limits and an explanation of why $\lim_{x \to 0} \frac{5}{x}$ does not exist. The ideas, though informally addressed in an Algebra 1 course, will expand students' thinking and reasoning skills. By comparing the answers obtained by using Bhaskara II's method with positive and negative numbers, students can easily see that they do not get a consistent value for $5 \div 0$. Since our argument for $5 \div 0$ can extend to any nonzero dividend, the result of any nonzero number divided by zero is undefined.

Also consider the ideas from above but using zero as the dividend. Using Bhaskara II's argument, we create the sequence $0 \div \frac{1}{10}, 0 \div \frac{1}{100}, 0 \div \frac{1}{1000}, \ldots$, in which every term is zero. As the divisor approaches 0, the quotient stays at 0. So we would imagine that $0 \div 0 = 0$. This result is different from infinity. Though this is a different limit from our previous examples, it is still the case that $0 \div 0$ is undefined. If we think about computing $0 \div 0$, we are attempting to find *x* such that $0 \div 0 = x$. This is equivalent to $0 \cdot x = 0$. Using the multiplication version produces a major issue since the value of *x* can be any real number. Since we have infinitely many solutions for *x*, we have infinitely many ways to define $0 \div 0$, making the quotient undefined.

Lesson Plan

Goals

- Gain concrete understanding of why division by zero is impossible.
- Introduce the idea of a limit.

Introduction

Ask the class why it is impossible to divide by zero. Discuss the student responses.

Lesson

Question: What is the value of $6 \div 3$? How do we know that the answer is 2?

- Division as Inverse of Multiplication: $6 \div 3 = 2$ because $2(3) = 6$.
- Division as Grouping: $6 \div 3 = 2$ because when we place 6 items into groups that each have 3 items, we create 2 groups. (See the diagram below).

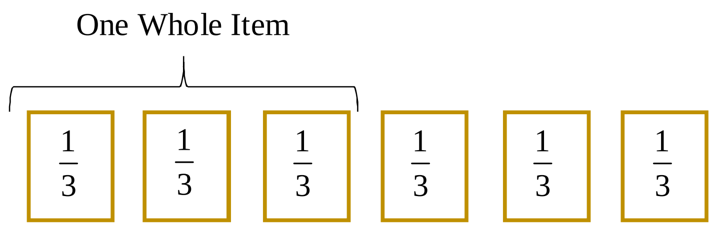
Question: We know that $2 \div \frac{1}{3} = 2(3) = 6$, but why is this correct?

Give students a chance to respond. Discuss these explanations:

- Division as Inverse of Multiplication: $2 \div \frac{1}{3} = x$ is equivalent to the following:

$$\frac{1}{3}(x) = 2 \;\Rightarrow\; \frac{x}{3} = 2 \;\Rightarrow\; 3\left(\frac{x}{3}\right) = 3(2) \;\Rightarrow\; x = 6.$$

- Division as Grouping: $2 \div \frac{1}{3}$ is the number of groups when we have a total of 2 items and each group has $\frac{1}{3}$ item. Since 2 items is 6 one-third items, the result is $2 \div \frac{1}{3} = 6$. (See the diagram below).



Annotated Student Worksheet

Consider the following sequence of fractions: $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \ldots$

Is this an increasing or decreasing sequence? **Decreasing**

★ What number is this sequence getting closer and closer to? **Zero**

Compute the following:

$$5 \div \frac{1}{10} = 50$$

$$5 \div \frac{1}{100} = 500$$

$$5 \div \frac{1}{1000} = 5000$$

$$5 \div \frac{1}{10000} = 50000$$

$$5 \div \frac{1}{100000} = 500000$$

$$5 \div \frac{1}{1000000} = 5000000$$

★ What do you notice about the quotients? Are the quotients getting close to a particular number?

The quotients are getting larger and larger. We say that they are heading towards infinity (which is not a number).

Conclusion

Examine the answers to the questions that are marked with the star symbols (★). Why do you think we say that dividing by zero is undefined? Explain however you can!

Example Answer: The closer the number you are dividing by is to zero, the larger the quotient will be. Therefore the quotients will be going towards infinity. The quotients will eventually be larger than any one particular number.

Extension

An ancient Indian mathematician, Bhaskara II, used the method above and determined that $1 \div 0 = \infty$. Was he correct? Give examples to defend your answer.

(See teacher guide.)

References and Further Reading

[1] Boyer, Carl B., and Merzback, Uta C. *A History of Mathematics*. John Wiley & Sons, Inc., 2011: pp. 186-202.

[5] Joseph, George G. *The Crest of the Peacock: Non-European Roots of Mathematics*. Princeton University Press, 2000: pp. 215-263.

[7] Mastin, Luke. “Indian Mathematics.” The Story of Mathematics (2010). <http://www.storyofmathematics.com/indian.html>.

3. COMPUTING SQUARE ROOTS: CHINESE

Background and Teacher Notes

Ancient Chinese mathematics was very much focused on solving real-world problems. The Chinese culture has many writings on mathematical topics but the text we are using for this particular lesson is one of the oldest in the culture. It is the *Chui-chang suan-shu*, also known as the *Jiuzhang suanshu* (*Nine Chapters on the Mathematical Art*). The fourth chapter of the *Nine Chapters* focuses on using mathematical procedures for land measurement and distribution. Many of the problems in this chapter focus on dividing up squares of land, which leads to the need to find square roots. This chapter provides the earliest evidence of a procedure for determining square roots in ancient Chinese mathematics.

In ancient times, Chinese mathematicians used counting rods, which were sticks that they laid out on the ground, a flat surface, or a counting board, and then manipulated. The *Nine Chapters* describes a process for computing square roots in terms of manipulations using counting rods, but we will be performing this process using modern notation. This method, though tedious, is a great alternative for students that does not require a calculator, but also is more efficient than just guess and check. It is popularly now called the Division Method.

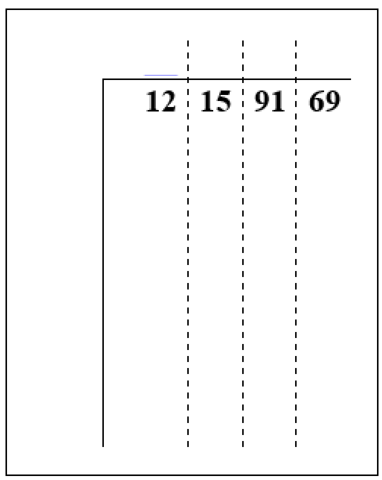
Students can use this process in the high school curriculum when computing the values of large square roots while working with quadratics. The lesson leads the class through the process and then has students compute square roots on their own. Since the process is a little tedious, direct instruction is required for the beginning. Then students will work independently to follow the procedure in several examples.

Teacher Guide

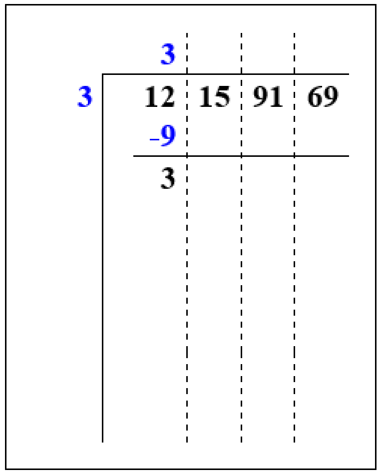
The example in this teacher guide is larger and more complex than any on the student worksheet. We provide this example in case students have complex questions about this process or for challenging students past this lesson.

Example: Find the square root of 12159169.

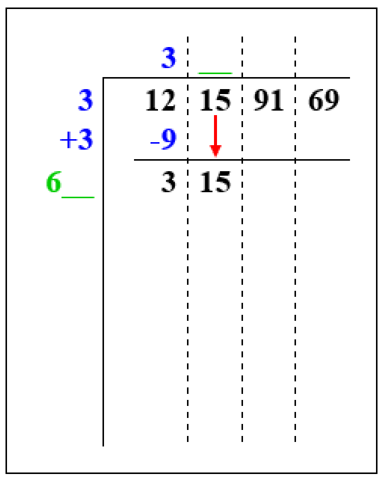
The setup is similar to that of a long division problem in modern American mathematics. Group the digits of the number in pairs, starting from the ones place and moving to the left.



Start with the left-most pair of numbers; in this example we start with 12. The first goal is to determine the largest whole number whose square is less than or equal to 12. The number would be 3 since $3^2 \leq 12 < 4^2$. This 3 is the value in the thousands place of $\sqrt{12159169}$. Place the number 3 both above and to the left of 12 in the diagram. Multiply the 3's together and subtract the product from 12. This leaves a difference of 3.



Bring down the next pair (15) and append it to the 3 in the first column, so the current number under consideration is 315. Add the 3 from the top of the first column to the 3 from the left of the first row to create 6 on the left hand side.



The next goal is to find the digit in the hundreds place of $\sqrt{12159169}$. We already know that $\sqrt{12159169}$ will have a 3 in the thousands place. Therefore $\sqrt{12159169} = 3000 + d$, for some $d < 1000$. If x is the digit in the hundreds place of $\sqrt{12159169}$, then x is in the ones place of $\frac{\sqrt{12159169}}{100}$.

Knowing that $\sqrt{12159169} = 3000 + d$, we can conclude that

$$\frac{\sqrt{12159169}}{100} = \frac{3000+d}{100} = 30 + \frac{d}{100}.$$

Since $d < 1000$, $\frac{d}{100} < 10$, so x is the whole-number part of $\frac{d}{100}$, and

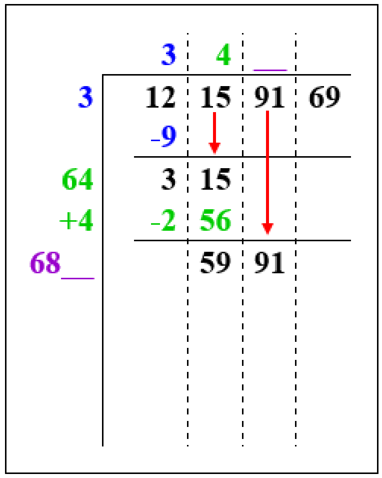
$$\frac{\sqrt{12159169}}{100} \approx 30 + x.$$

Squaring both sides, we get $\frac{12159169}{10000} \approx 900 + 60x + x^2$. Still considering only the whole number parts, this gives $1215 \approx 900 + 60x + x^2$, and thus,

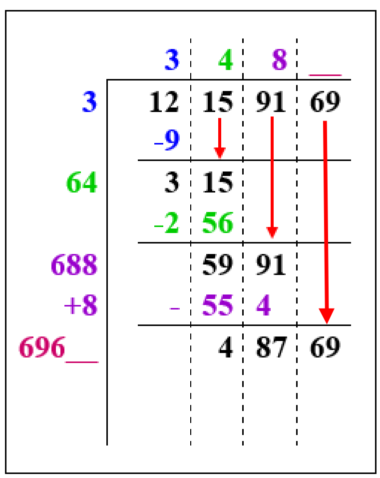
$$315 \approx 60x + x^2 = (60+x)x.$$

This explains why we find the next digit (the hundreds digit) of $\sqrt{12159169}$ by multiplying the 6 in the left column by 10 and finding the largest digit *x* so that $(60+x)(x) \le 315$. The hundreds digit will be 4 because $64(4) \le 315 < 65(5)$.

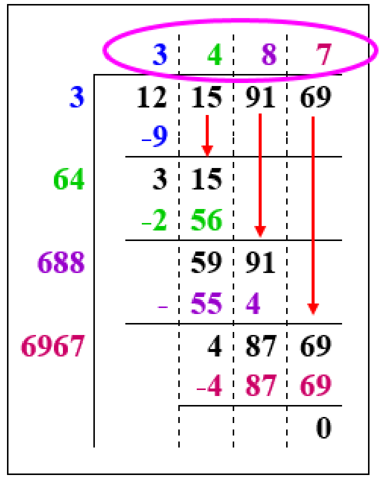
As shown below we repeat these steps in each successive column: Place the 4 from the last step at the top of the second column and in the ones position of 64 on the left side. Multiply the 64 at the left by the 4 at the top. Subtract the product $(64)(4) = 256$ from 315. Since the difference is not 0, bring down the next pair of numbers. Add the 4 from the top of the second column to the 64 at the left to create 68 on the outside left.



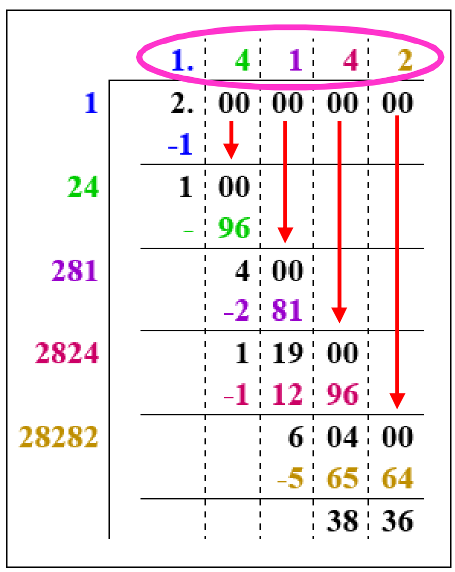
Determine next what the tens digit of the solution must be: multiply the 68 by 10, and find the largest whole number *x* such that $(680+x)(x)\le 5991$. The tens digit is 8 because $(688)(8)\le 5991<(689)(9)$. Record the 8 at the top and as the ones digit of 688 in the left column. Subtract $688(8)$ from 5991. Since the difference is not 0, bring down the next pair of digits of 12159169, and add $688+8$ in the left column. Now repeat the earlier steps to determine the next digit in $\sqrt{12159169}$, which happens to be the ones digit. The ones digit is 7 because $6967(7)\le 48769<6968(8)$.



Subtracting the product $6967(7)$ gives zero, so $\sqrt{12159169}$ is exactly 3487.



All of the examples and problems on the student worksheet involve perfect squares. If we start with a number that is not a perfect square, we extend the process by appending pairs of zeros after the decimal. We can continue until we achieve our desired level of accuracy. See the example below for $\sqrt{2}$.



Lesson Plan

<u>Goals</u>

Compute square roots without a calculator.

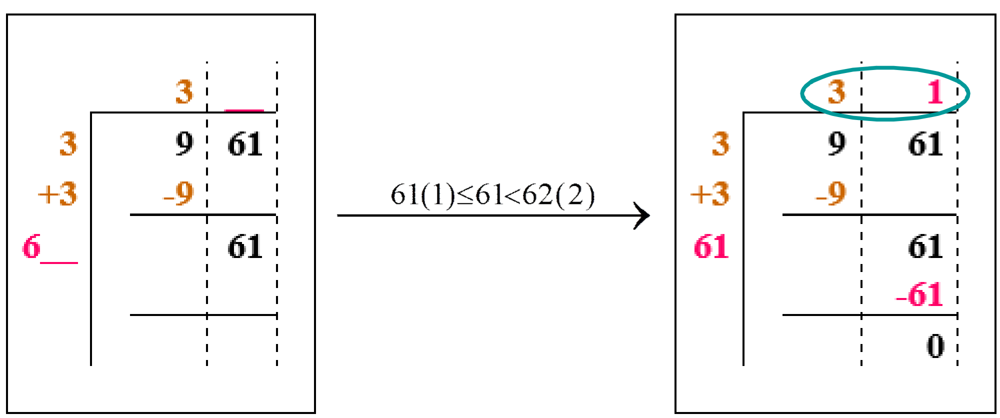
<u>Introduction</u>

Ask students to pretend they lived back in the days of ancient cultures and they are land surveyors. They have their current mathematical knowledge but do not have modern technology. To complete their duties, they need to be able to find the value of $\sqrt{961}$. Ask students for suggestions for doing this computation.

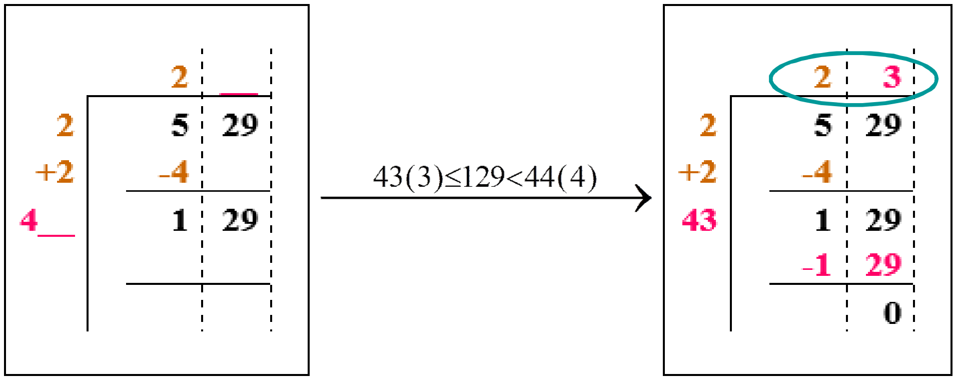
After discussing the students' suggestions, work through the first three examples together as a class.

<u>Annotated Student Worksheet</u>

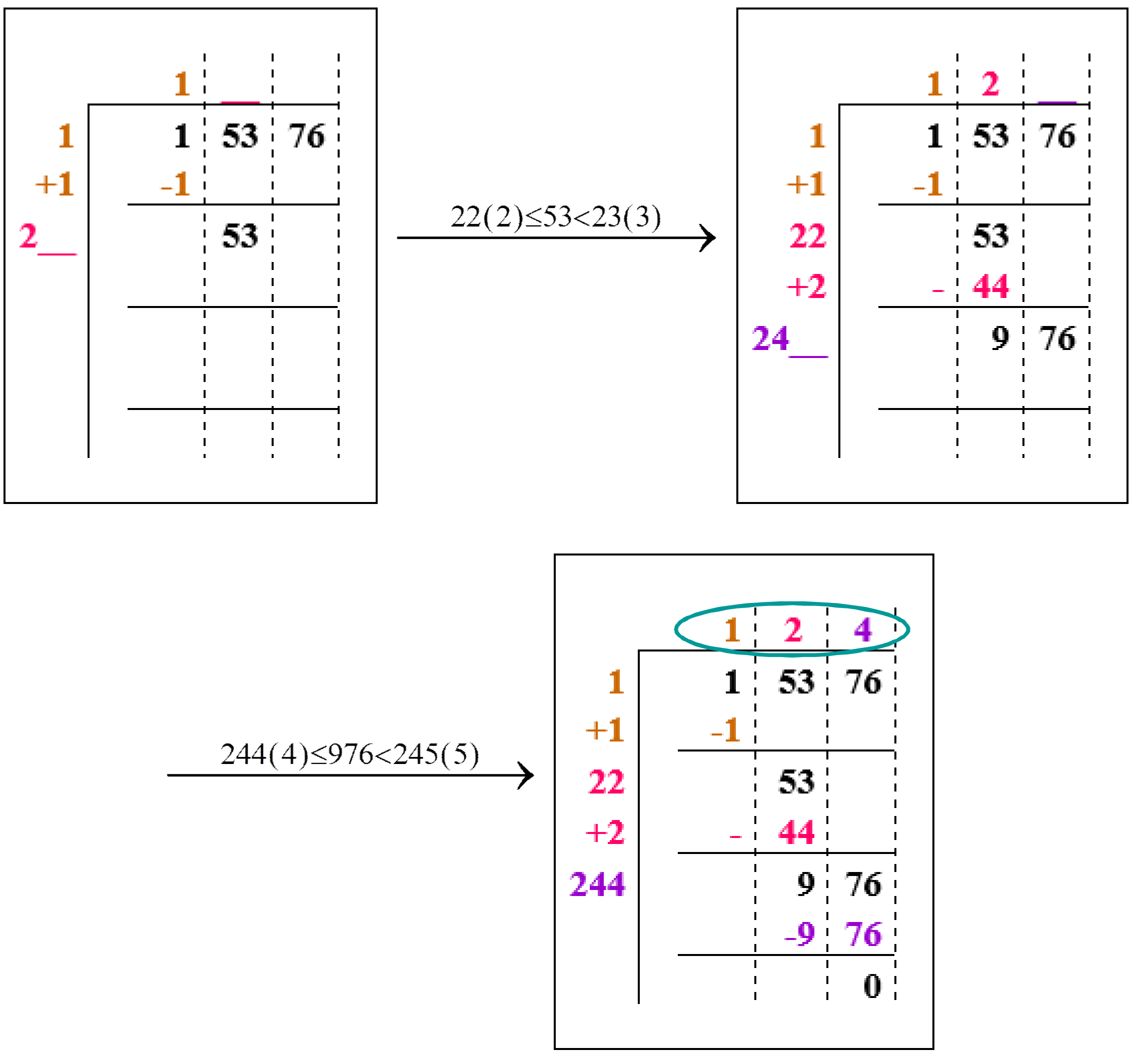
Example: Compute $\sqrt{961}$.



Example: Compute $\sqrt{529}$.



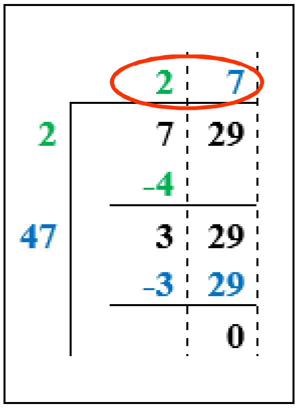
Example: Compute $\sqrt{15376}$.



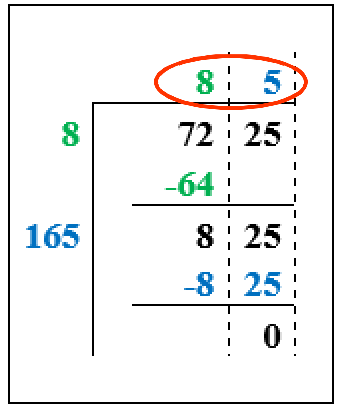
Worktime

Work these problems on your own. Show all work by hand! Check your answers.

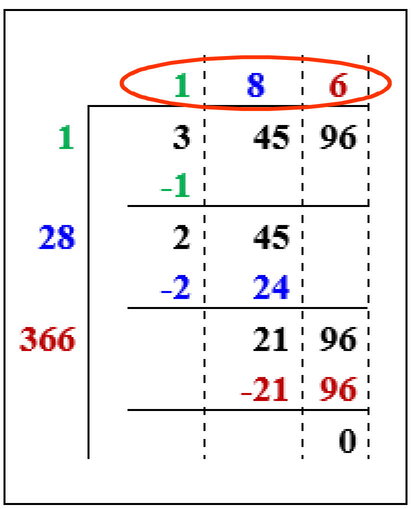
1. $\sqrt{729}$



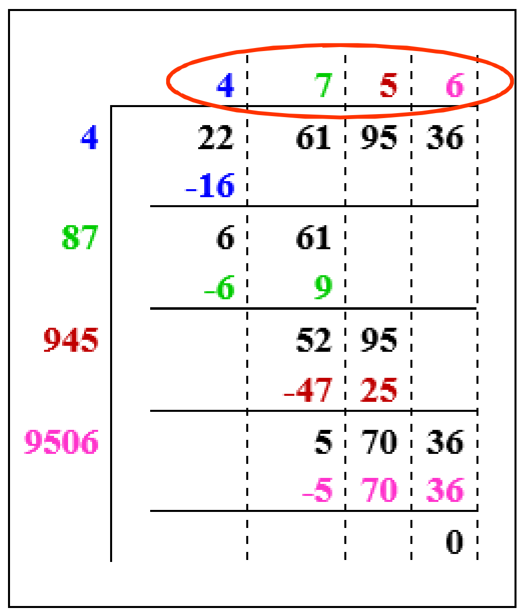
2. $\sqrt{7225}$



3. $\sqrt{34596}$



4. Challenge Problem! $\sqrt{22619536}$



References and Further Reading

[1] Boyer, Carl B., and Merzback, Uta C. *A History of Mathematics*. John Wiley & Sons, Inc., 2011: pp. 175-185.

[5] Joseph, George G. *The Crest of the Peacock: Non-European Roots of Mathematics*. Princeton University Press, 2000: pp. 130-177.

[6] Dauben, Joseph W., "Chinese Mathematics." In *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook*, Victor J. Katz [ed.]. Princeton University Press, 2007: pp. 187-204.

[8] Nelson, David, and Joseph, George G., and Williams, Julian. *Multicultural Mathematics: Teaching Mathematics from a Global Perspective*. Oxford University Press, 1993: pp. 55-57.

4. SOLVING QUADRATIC EQUATIONS: ISLAMIC

Background and Teacher Notes

The spread of Islam throughout different countries had a great influence on the development and spread of mathematics. This is evident from the number of countries where ancient Islamic mathematicians worked from about 750 to 1450 CE. We consider the geographical region associated with "Islamic" mathematics to be the Iberian Peninsula through North Africa, the Middle East, the central Asian republics of the former Soviet Union, Afghanistan, Iran, and sections of India [6, p. 515]. Sometimes Islamic mathematics is called Arab mathematics, but *Islamic* is a more appropriate name, since many of the mathematicians were not necessarily Arabic, though they were Islamic. The spread of Islam was due to the Prophet Muhammad and his followers' preaching of faith. That created many enemies for the Islamic culture and led to a period of time filled with many Arabic conquests, lasting over a century. Through this spread, the culture expanded greatly and Baghdad became a major hub of development in science and mathematics. Much of modern day school mathematics, especially algebra, can be traced back to Baghdad and the mathematicians who worked there.

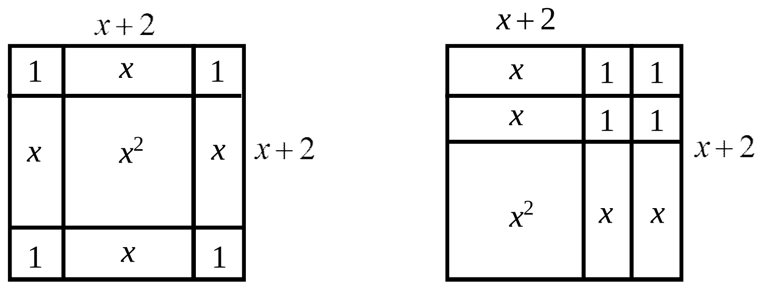
One of the greatest Islamic mathematicians, Muhammad ibn Musa al-Khwarizmi, was a talented scientist who eventually moved to Baghdad and became one of the most popular mathematicians working with arithmetic and most importantly, what we now call algebra. He wrote many books and can be considered the father of algebra; the word *algorithm* is literally derived from his name.

The technique of completing the square, as it is now called, is credited to Islamic mathematicians. Specifically, teachers should give credit to al-Khwarizmi, who used this method to solve quadratics in the form $x^2 + bx = c$. The method in this lesson creates a deeper understanding of solving quadratics because it includes a visual exploration along with the algebraic steps. The geometric description gives different learners a new opportunity to make connections and succeed.

This lesson is meant to be an introduction to the process of completing the square. It is intended to be a project-based lesson that uses technology to make connections. After the class completes the introductory problem and the discussion in a large group, students work independently or in small groups without the teacher to complete the lesson project for this topic. This lesson directs students to use interactive algebra tiles online, but physical algebra tiles, other online algebra tiles, or mobile apps can be used.

Teacher Guide

Present a quadratic equation to solve; for example $x^2 + 4x = 13$. Lead students through the process of completing the square. Students will be using algebra tiles to build squares. Two completed squares for this example are shown below with their dimensions. Refer to the explanation in the Lesson Plan for information on these algebra tiles.



In solving $x^2 + 4x = 13$, we can use the area of either square above as follows:
In either diagram, the area of the large square is $(x+2)^2$, so by comparing areas, we see that $(x+2)^2 - 4(1) = x^2 + 4x$. So for the equation $x^2 + 4x = 13$, we get $(x+2)^2 - 4 = 13$. Thus, $(x+2)^2 = 17$, and so $x = -2 \pm \sqrt{17}$.

Lesson Plan

Goals

Build the idea of completing the square to find algebraic shortcuts.

Introduction

Start the lesson with the following equations:

$$(x+1)^2 = 4 \quad \text{and} \quad x^2 + 4x = 5.$$

Ask students to solve both. Students will most likely solve the first equation by taking square roots and the second equation by factoring.

Discussion

Ask the students:

- Are both equations quadratics?
- Was one quadratic easier to solve than the other, and why?
- What if it wasn't obvious that factoring was possible?

Leading Question

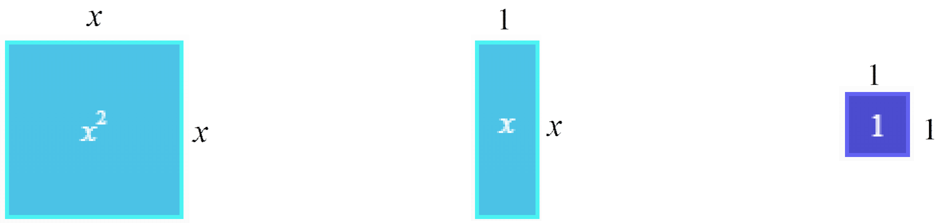
Is there an equation that is equivalent to the second example, but in the form of the first example? We are going to use modern technology to investigate this question.

Annotated Student Worksheet

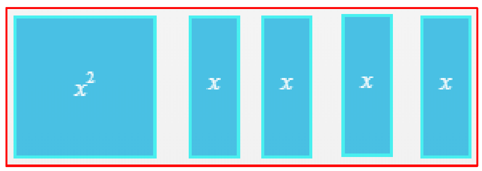
Consider this equation again: $x^2 + 4x = 5$.

We will be using virtual algebra tiles at https://technology.cpm.org/general/tiles/ to help change the form of this quadratic. The three types of pieces are shown below with their dimensions labeled.

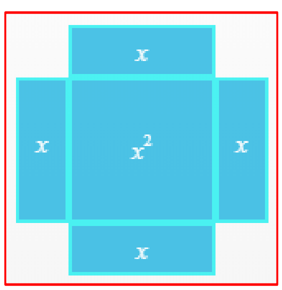
x^2 Piece **x Piece (can be rotated)** **Unit Piece**



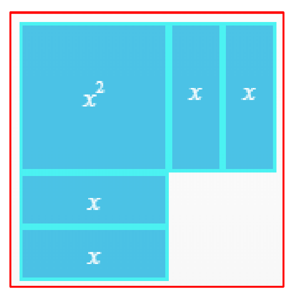
1. Drag the pieces needed to create $x^2 + 4x$ into the workspace.



2. Arrange the pieces so that they almost form a square. There may be more than one configuration. If you need to rotate one of the pieces, double click the piece.

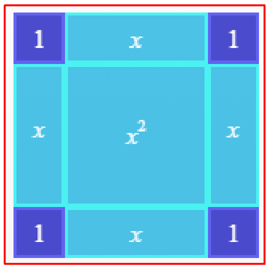


Should resemble one of these two configurations, possibly rotated.

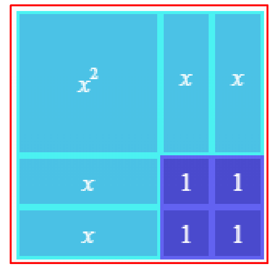


Were you able to create a perfect square? **No**

3. Add any other pieces needed to create a square from your existing arrangement. Use as few pieces as possible.



Should resemble one of these two configurations, possibly rotated.



What pieces (and how many) did you need to add in order to make a perfect square?

4 unit pieces

4. What are the length and width (base and height) of the finished square?

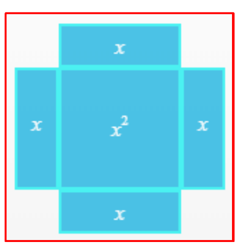
Both are $x+2$

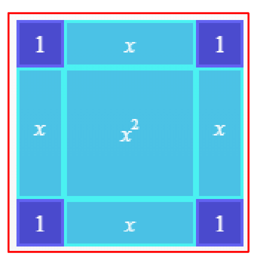
5. What is the area of the square? $\mathbf{A=(x+2)^2}$

6. Look back at the original equation: $x^2+4x=5$

Be careful to not change the equation. Is it true that $(x+2)^2=x^2+4x$? Why?

No! Either expand $(x+2)^2$ to show $x^2+4x+4\neq x^2+4x$, or compare the two shapes.





Notice that the full square with area x^2+4x+4 is 4 units larger than the non-square whose area is x^2+4x.

Fill in the missing boxes to make true statements.

$$x^2 + 4x + 4 + \boxed{-4} = x^2 + 4x$$

$$(x+2)^2 + \boxed{-4} = x^2 + 4x$$

So $x^2 + 4x = 5$ is equivalent to $\underline{(x+2)^2 - 4} = 5$.

7. Now solve the new form of the equation! Do you get the same solutions we got during the class discussion?

$$\begin{aligned} x^2 + 4x &= 5 \\ (x+2)^2 - 4 &= 5 \\ (x+2)^2 &= 9 \\ x+2 &= \pm\sqrt{9} \\ x+2 &= \pm 3 \\ x &= \pm 3 - 2 \\ x &= -5, 1 \end{aligned}$$

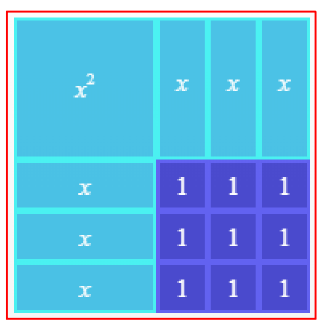
This method is called **completing the square**.

Try this method with two more quadratic equations!

$x^2 + 6x = -8$

$(x+3)^2 - 9 = -8$

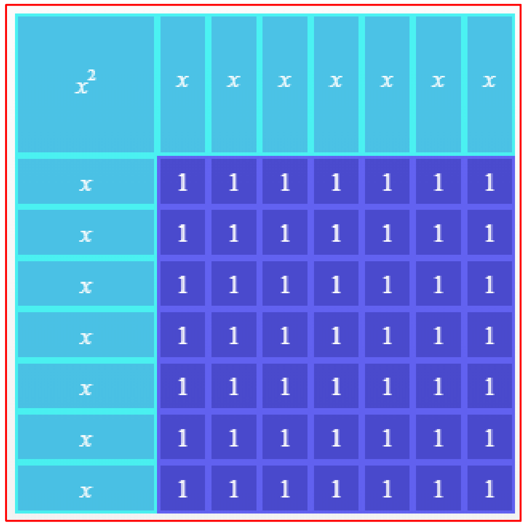
$x = -4, -2$



$x^2 + 14x = 38$

$(x+7)^2 - 49 = 38$

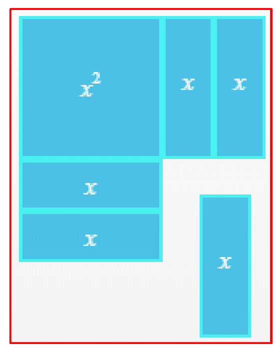
$x = -7 \pm \sqrt{87}$



<u>Bonus</u>

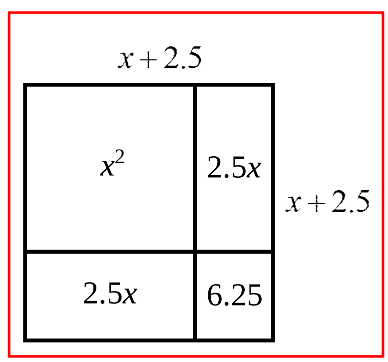
- What happens when you try to use algebra tiles to solve $x^2 + 5x = 10$?

  We cannot physically build a square because we would need to split the fifth *x* piece in half, meaning we would need 2.5 *x* pieces on each side.



- Does this mean we cannot complete the square?

  No. We just cannot build the square with physical algebra tiles. We can use the same idea and imagine that we have 2.5 of the "*x*" pieces on each side.



References and Further Reading

[1] Boyer, Carl B., and Merzback, Uta C. *A History of Mathematics*. John Wiley & Sons, Inc., 2011: pp. 203-211.

[2] Center for South Asian & Middle Eastern Studies. "Islamic Mathematics." University of Illinois at Urbana-Champaign. <http://www.csames.illinois.edu/documents/outreach/Islamic_Mathematics.pdf>.

[5] Joseph, George G. *The Crest of the Peacock: Non-European Roots of Mathematics*. Princeton University Press, 2000: pp. 301-333.

[6] Berggren, J. L., "Mathematics in Medieval Islam." In *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook*, Victor J. Katz [ed.]. Princeton University Press, 2007: pp. 515-520, 542-560.

[9] Rashed, Roshdi. *Al-Khwarizmi: The Beginnings of Algebra*. Saqi, 2009: pp. 3-34.

5. SOLVING THIRD-DEGREE EQUATIONS: ISLAMIC

Background and Teacher Notes

In this activity we describe a particular method for approximating solutions of cubic functions, in which the solution is the intersection point of two conic sections. This method was developed by the Islamic mathematician Omar Khayyam, who used geometry to solve cubic equations. Khayyam wrote his own book, *Algebra*, in which he developed the work of al-Khwarizmi and described methodology for approximating roots of various types of third-degree polynomials. Greek influence is obvious in this technique due to the geometry involved. This lesson is a compilation of algebra and geometry, which is extremely common in the high school curriculum.

Islamic mathematicians did not use the modern technology used in this lesson, but the process is identical. We will identify special lines and points for students but these can be constructed relative to the conics without using coordinate geometry. Some students may wonder why Omar Khayyam used this particular method instead of directly solving algebraically. That answer is simple; solving directly is either extremely difficult or impossible. Even most modern techniques result in an approximated solution. It was much more efficient to construct the conics and then use measurement tools to find an extremely close approximation to their intersection. The concept of using coordinate axes to plot the points of an equation was nonexistent in the time of Omar Khayyam, so estimating roots from the graph of a cubic was not an option.

The lesson for this topic is designed as a project that students can do individually or in small groups, but with little teacher intervention. It is an alternative look at approximating roots of a specific type of cubic equation and should be used along with other methods to solve for the real roots. Even though we are more likely today to approximate with something like Newton's Method, using conic sections entwines algebra and geometry in this process. We have written the activity assuming that students will be using GeoGebra, but it can be adapted for use with any dynamic geometry platform. The lesson walks the students through each step and is designed for students who are familiar with solving equations and have experience using GeoGebra. For students to be successful with this project, they must be able complete the following tasks in GeoGebra:

- Graph equations and plot points
- Create a line segment between points
- Create a circle with center through point
- Place intersection point
- Create a perpendicular line through a point
- Calculate the distance between two points

Teacher Guide

To solve third-degree equations in the form $x^3 + a^2x = b$, we use the parabola $x^2 = ay$ and a circle with a diameter of $\frac{b}{a^2}$ whose center is $\left(\frac{b}{2a^2}, 0\right)$. These conics will intersect at two points. The root of the cubic is the *x*-coordinate of the point of intersection that is not the vertex of the parabola. We verify this as follows:

The desired intersection of the circle and parabola will be the point (x_0, y_0) where $x_0 \neq 0$. This means that $x_0{}^2 = ay_0$ and $\left(\frac{b}{2a^2}\right)^2 = \left(x_0 - \frac{b}{2a^2}\right)^2 + y_0{}^2$.

Then

$$\left(\frac{b}{2a^2}\right)^2 = \left(x_0 - \frac{b}{2a^2}\right)^2 + \left(\frac{x_0{}^2}{a}\right)^2,$$

so

$$\frac{b^2}{4a^4} = x_0{}^2 - 2\left(\frac{bx_0}{2a^2}\right) + \frac{b^2}{4a^4} + \frac{x_0{}^4}{a^2},$$

and thus

$$0 = x_0{}^2 - \frac{bx_0}{a^2} + \frac{x_0{}^4}{a^2}.$$

Then

$$0 = a^2x_0{}^2 - bx_0 + x_0{}^4 = x_0\left(a^2x_0 - b + x_0{}^3\right).$$

Since $x_0 \neq 0$, we have $a^2x_0 - b + x_0{}^3 = 0$, and thus $x_0{}^3 + a^2x_0 = b$. So x_0 is a root of the cubic equation.

The type of cubic equation used in this method always has just one solution. We can demonstrate this with calculus. The function $f(x) = x^3 + a^2x - b$ has derivative

$f'(x) = 3x^2 + a^2$. Since $a \neq 0$, $f'(x) > 0$, and thus $f(x)$ will always increase. Hence there will only be one intersection of $f(x)$ and the *x*-axis.

In the student worksheet, students choose their own equation of the form $x^3 + a^2x = b$, where *a* and *b* are both nonzero. We have shown the student worksheet with answers using the equation $x^3 + 4x = 10$.

Lesson Plan

Tell the students that they are going to follow a procedure for solving a particular type of cubic equation, using a method that was developed by Omar Khayyam. Refer to the Background and Teacher Notes for more information.

Annotated Student Worksheet

Find a real solution to a cubic equation of the form $x^3 + a^2 x = b$.

- Use GeoGebra or another dynamic platform.
- Choose your own cubic equation in the form $x^3 + a^2 x = b$, where *a* and *b* are both nonzero.

  Fill in your choice: $x^3 + (2)^2 x = (10)$

  This means $a = $ 2 $b = $ 10

To solve this problem, you will be graphing a quadratic equation and a circle.

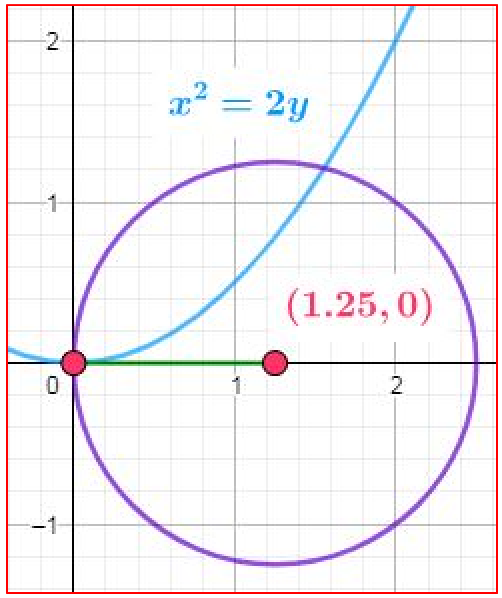
Quadratic

- Graph the quadratic equation in the form of $x^2 = ay$, using your value of *a*.

  Your Quadratic: $x^2 = (2)y$

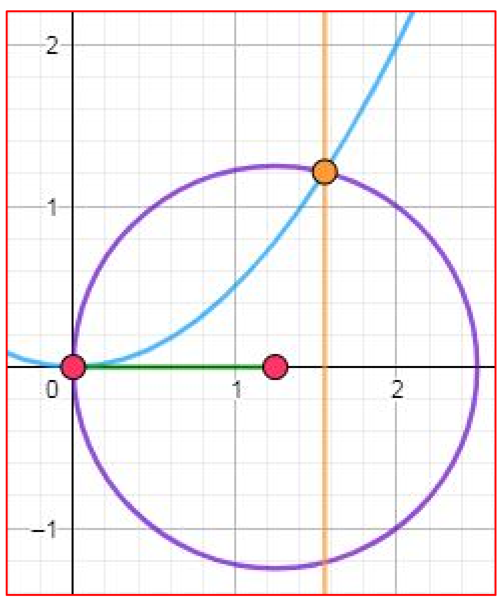
Circle

- Create the line segment that represents the radius of the circle as follows:
  - Place the endpoints at $(0,0)$ and $\left(\frac{b}{2a^2}, 0\right)$.
  - Create a circle with center at $\left(\frac{b}{2a^2}, 0\right)$ so that the circle includes the point $(0,0)$.

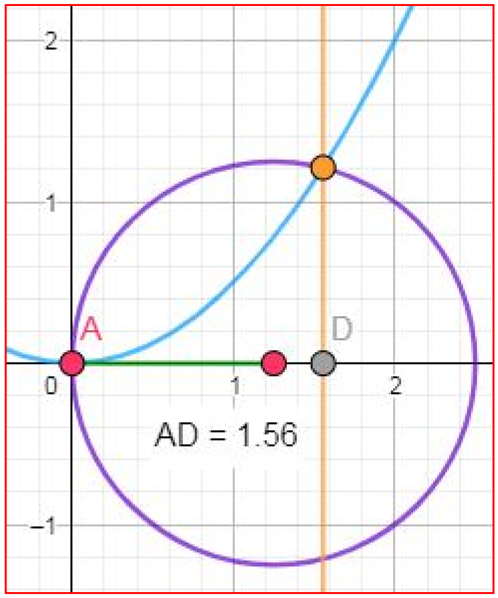


Finding the Solution

◦ Mark the intersection point of the parabola and the circle, and create a line through the intersection point that is perpendicular to the *x*-axis.



◦ Mark the intersection point of the vertical line and the *x*-axis. The distance from the origin to that point will be the solution to the cubic equation.



My solution is: $x \approx \mathbf{1.56}$

Tasks

1. Print a screenshot of your GeoGebra worksheet and attach it to this worksheet.

2. Verify that the answer you found is correct.

$$(1.56)^3 + 4(1.56) = 10.036416 \approx 10$$

*This method allows for negative values of *b*. The only difference will be that the center of the circle is on the negative *x*-axis.

References and Further Reading

[1] Boyer, Carl B., and Merzback, Uta C. *A History of Mathematics*. John Wiley & Sons, Inc., 2011: pp. 218-220.

[4] Henderson, David W. *Experiencing Geometry: Euclidean and non-Euclidean with History, 3rd Edition.* Pearson Prentice Hall, 2005: pp. 272-279.

[5] Joseph, George G. *The Crest of the Peacock: Non-European Roots of Mathematics*. Princeton University Press, 2000: pp. 309-310, 328-332.

[6] Berggren, J. L., "Mathematics in Medieval Islam." In *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook*, Victor J. Katz [ed.]. Princeton University Press, 2007: pp. 556-560.

6. FUN WITH ROOTS: ISLAMIC

Background and Teacher Notes

The Islamic mathematician Muhammad ibn Musa al-Khwarizmi wrote a treatise on algebra around the year 820 called *Book of Algebra* and was the major influencer for algebraic Arab work until his death in 850. In the introduction, he states that his reason for creating his treatise was to “compose a concise book on the form of calculation in algebra [9, p. 24].” Al-Khwarizmi wrote this book for people who needed algebraic calculations in order to survey, dig, and change the land around them.

This lesson involves the description of a process for computing multiples of the square root of a number. It is not meant to be a method for students to use on a regular basis, because the mathematics has advanced from the time of al-Khwarizmi. Rather, the point of this activity is for students to explain why this method works and whether it works for all cases. Ultimately, this leads students to use knowledge of simplifying radicals.

This activity is meant to be an extension activity or homework project. Students need to first understand the vocabulary for working with roots and then find a way to prove that the procedure is valid. This does not need to be any sort of formal proof, but the teacher should push students to find a way to verify that this procedure works in general, rather than relying on specific examples. A discussion of the process for irrational numbers and/or negative numbers can be included.

Teacher Guide

Goal

Perform multiplication on the root of a nonnegative number *N*, by a positive multiplier, *m*. That is, compute $m\sqrt{N}$.

Process

- Find the value *V*, where $V = (m)(m)(N)$.
- The desired product is the square root of *V*. That is, $m\sqrt{N} = \sqrt{V}$.

Proof

For any nonnegative *N* and positive number *m*, let $V = m^2 N$. Then,

$$\sqrt{V} = \sqrt{m^2 N} = \sqrt{m^2}\sqrt{N} = |m|\sqrt{N} = m\sqrt{N} .$$

The questions in the *Book of Algebra* are phrased in terms of multipliers that double, triple, or halve the square root of a number, which are all positive numerical values.

Notice that if *m* is negative, then $|m|\sqrt{N} \neq m\sqrt{N}$, so this method does not work. This is a really good opportunity to discuss with students why $\sqrt{x^2} = x$ in general.

Extension

We can perform multiplication on the n^{th} root of a number *N* similarly. That is, we can compute $m\sqrt[n]{N}$ for any positive *m* and $N \in \mathbb{R}$ for which $\sqrt[n]{N}$ is defined: for $V = m^n N$, $\sqrt[n]{V} = \sqrt[n]{m^n N} = \sqrt[n]{m^n}\sqrt[n]{N} = m\sqrt[n]{N}$. The only concern is to make sure that *N* is nonnegative if *n* is even.

Lesson Plan

Goals

- Deepen vocabulary and understanding of roots.
- Translate verbal descriptions into mathematical processes and notation.
- Prove why this process works to the best of the student's ability.

Annotated Student Worksheet

1a). What number is double the square root of 16? **8**

1b). What number is triple the square root of 64? **24**

1c). What number is half of the square root of 100? **5**

How did you get these answers?

Let's take a look at another way to get these values.

Double the Root of 16

2a). When we double a number, we are multiplying by: **2**

2b). Multiply 16 by the value from 2a) twice: $(2)(2)(16) = \boxed{64}$

2c). Take the square root of the product from 2b): $\sqrt{64} = \boxed{8}$

2d). Compare the value from 2(c) to the answer to Question #1. Are they the same? **yes**

Triple the Root of 64

3a). When we triple a number, we are multiplying by: **3**

3b). Multiply 64 by the value from 3a) twice: $(3)(3)(64) = \boxed{576}$

3c). Take the square root of the product from 3b): $\sqrt{576} = \boxed{24}$

3d). Compare the value from 3c) to the answer to Question #2. Are they the same? **yes**

Halve the Root of 100

4a). When we cut a number in half, we are multiplying by: **0.5**

4b). Multiply 100 by the value from 4a) twice: $(0.5)(0.5)(100) = \boxed{25}$

4c). Take the square root of the product from 4b): $\sqrt{25} = \boxed{5}$

4d). Compare the value from 4c) to the answer to Question #3. Are they the same? **yes**

5.) Create three more questions similar to Questions 2-4. Solve each of your problems using the process described above. Does the procedure work for all of your examples? Pick some values that are not perfect squares, or are decimals or fractions.

Student Answers

Conclusion

6.) Why does this method work? Do your best to prove why for ALL cases.

Students will have various responses that should

have some version of what is in the Teacher Guide.

References and Further Reading

[5] Joseph, George G. *The Crest of the Peacock: Non-European Roots of Mathematics*. Princeton University Press, 2000: pp. 301-307.

[9] Rashed, Roshdi. *Al-Khwarizmi: The Beginnings of Algebra*. Saqi, 2009: pp. 3-25, 132.

7. Student Worksheet Templates

This section contains the blank student worksheets that accompany each lesson. Teachers are free to use, share, or change these lesson templates appropriately for students.

- Method of False Position: Egyptian
- Division by Zero: Indian
- Computing Square Roots: Chinese
- Solving Third-Degree Equations: Islamic
- Solving Quadratic Equations: Islamic
- Fun with Roots: Islamic

Method of False Position: Egyptian **Name:**

1. Solve the following problems using the Method of False Position. Show all work!

a) Problem 25 from the Ahmes Papyrus
A quantity and its $\frac{1}{2}$ added together become 16. What is the quantity?

b) Problem 27 from the Ahmes Papyrus
A quantity and its $\frac{1}{5}$ added together become 21. What is the quantity?

c) Solve for *x*: $\frac{1}{3}x+\frac{1}{5}x+x-\frac{1}{2}\left(x+\frac{1}{3}x\right)=5$

d) Solve for *x*: $x+\frac{2}{3}x+10=110$

e) Two-sevenths of a quantity is subtracted from its double and together become 80. What is the quantity?

2. Give some reasons why, or situations when, the Method of False Position would be more efficient or better than modern algebraic methods.

Consider the following sequence of fractions: $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \ldots$

Is this an increasing or decreasing sequence?

★ What number is this sequence getting closer and closer to?

Compute the following:

$$5 \div \frac{1}{10} =$$

$$5 \div \frac{1}{100} =$$

$$5 \div \frac{1}{1000} =$$

$$5 \div \frac{1}{10000} =$$

$$5 \div \frac{1}{100000} =$$

$$5 \div \frac{1}{1000000} =$$

★ What do you notice about the quotients?
Are the quotients getting close to a particular number?

Conclusion

Examine the answers to the questions that are marked with the star symbols (★). Why do you think we say that dividing by zero is undefined? Explain however you can!

Extension

An ancient Indian mathematician, Bhaskara II, used the method above and determined that $1 \div 0 = \infty$. Was he correct? Give examples to defend your answer.

Computing Square Roots: Chinese **Name:**

Example: Compute $\sqrt{961}$.

Example: Compute $\sqrt{529}$.

Example: Compute $\sqrt{15376}$.

Worktime

Work these problems on your own. Show all work by hand! Check your answers.

1. $\sqrt{729}$

2. $\sqrt{7225}$

3. $\sqrt{34596}$

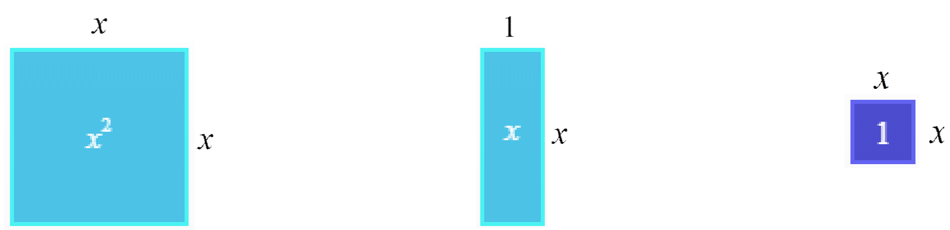
4. Challenge Problem! $\sqrt{22619536}$

Solving Quadratic Equations: Islamic **Name:** ______________________

Consider this equation again: $x^2 + 4x = 5$.

We will be using virtual algebra tiles at https://technology.cpm.org/general/tiles/ to help change the form of this quadratic. The three types of pieces are shown below with their dimensions labeled.

x^2 Piece **x Piece (can be rotated)** **Unit Piece**



1. Drag the pieces needed to create $x^2 + 4x$ into the workspace.

2. Arrange the pieces so that they almost form a square. There may be more than one configuration. If you need to rotate one of the pieces, double click the piece.

   Were you able to create a perfect square?

3. Add any other pieces needed to create a square from your existing arrangement. Use as few pieces as possible.

   What pieces (and how many) did you need to add in order to make a perfect square?

4. What are the length and width (base and height) of the finished square?

5. What is the area of the square?

6. Look back at the original equation: $x^2 + 4x = 5$

Be careful to not change the equation. Is it true that $(x+2)^2 = x^2 + 4x$? Why?

Notice that the full square with area $x^2 + 4x + 4$ is 4 units larger than the non-square whose area is $x^2 + 4x$.

Fill in the missing boxes to make true statements.

$$x^2 + 4x + 4 + \square = x^2 + 4x$$

$$(x+2)^2 + \square = x^2 + 4x$$

So $x^2 + 4x = 5$ is equivalent to _________ $= 5$

7. Now solve the new form of the equation! Do you get the same solutions we got during the class discussion?

This method is called <u>**completing the square**</u>.

Try this method with two more quadratic equations!

$x^2 + 6x = -8$

$x^2 + 14x = 38$

<u>Bonus</u>

- What happens when you try to use algebra tiles to solve $x^2 + 5x = 10$?

- Does this mean we cannot complete the square?

Find a real solution to a cubic equation of the form $x^3 + a^2x = b$.

- Use GeoGebra or another dynamic platform.
- Choose your own cubic equation in the form $x^3 + a^2x = b$, where *a* and *b* are both nonzero.

Fill in your choice: $x^3 + (\quad)^2 x = (\quad)$

This means $a =$ ____ $b =$ ____

To solve this problem, you will be graphing a quadratic equation and a circle.

Quadratic

- Graph the quadratic equation in the form of $x^2 = ay$, using your value of *a*.

Your Quadratic: $x^2 = (\quad)y$

Circle

- Create the line segment that represents the radius of the circle as follows:
  - Place the endpoints at $(0,0)$ and $\left(\frac{b}{2a^2}, 0\right)$.
  - Create a circle with center at $\left(\frac{b}{2a^2}, 0\right)$ so that the circle includes the point $(0,0)$.

Finding the Solution

◦ Mark the intersection point of the parabola and the circle, and create a line through the intersection point that is perpendicular to the x-axis.

◦ Mark the intersection point of the vertical line and the x-axis. The distance from the origin to that point will be the solution to the cubic equation.

My solution is: [ ]

Tasks

3. Print a screenshot of your GeoGebra worksheet and attach it to this worksheet.

4. Verify that the answer you found is correct.

Fun with Roots: Islamic **Name:** ____________________

1a). What number is double the square root of 16? ______

1b). What number is triple the square root of 64? ______

1c). What number is half of the square root of 100? ______

How did you get these answers?

Let's take a look at another way to get these values.

Double the Root of 16

2a). When we double a number, we are multiplying by: ______

2b). Multiply 16 by the value from 2a) twice: $(\quad)(\quad)(16) = \boxed{}$

2c). Take the square root of the product from 2b): $\sqrt{} = \boxed{}$

2d). Compare the value from 2(c) to the answer to Question #1. Are they the same?

Triple the Root of 64

3a). When we triple a number, we are multiplying by: ______

3b). Multiply 64 by the value from 3a) twice: $(\quad)(\quad)(64) = \boxed{}$

3c). Take the square root of the product from 3b): $\sqrt{\quad} = \boxed{\quad}$

3d). Compare the value from 3c) to the answer to Question #2. Are they the same?

Halve the Root of 100

4a). When we cut a number in half, we are multiplying by: ______

4b). Multiply 100 by the value from 4a) twice: $(\quad)(\quad)(100) = \boxed{\quad}$

4c). Take the square root of the product from 4b): $\sqrt{\quad} = \boxed{\quad}$

4d). Compare the value from 4c) to the answer to Question #3. Are they the same?

5.) Create three more questions similar to Questions 2-4. Solve each of your problems using the process described above. Does the procedure work for all of your examples? Pick some values that are not perfect squares, or are decimals or fractions.

Conclusion

6.) Why does this method work? Do your best to prove why for ALL cases.

8. OHIO'S LEARNING STANDARDS FOR MATHEMATICS (REV. 2017)

Listed below are specific standards applicable to the lessons provided in this work.

A.APR.1 Understand that polynomials form a system analogous to the integers, namely, that they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials. a. Focus on polynomial expressions that simplify to forms that are linear or quadratic.

Applicable Lessons Number(s): 1, 4

A.APR.3 Identify zeros of polynomials, when factoring is reasonable, and use the zeros to construct a rough graph of the function defined by the polynomial.

Applicable Lessons Number(s): 3, 4, 5

A.REI.3 Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

Applicable Lessons Number(s): 1

A.REI.4 Solve quadratic equations in one variable. a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x-p)^2 = q$ that has the same solutions. b. Solve quadratic equations as appropriate to the initial form of the equation by inspection, e.g., for $x^2 = 49$; taking square roots; completing the square; applying the quadratic formula; or utilizing the Zero-Product Property after factoring.

Applicable Lessons Number(s): 3, 4, 6

F.BF.2 Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.

Applicable Lessons Number(s): 2

F.IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. a. Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

Applicable Lessons Number(s): 3, 4, 5, 6

G.GMD.1 Give an informal argument for the formulas for the circumference of a circle, area of a circle, and volume of a cylinder, pyramid, and cone. Use dissection arguments, Cavalieri's principle, and informal limit arguments.

Applicable Lessons Number(s): 2

9. COMPLETE LIST OF REFERENCES AND FURTHER READING

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[4] Henderson, David W. *Experiencing Geometry: Euclidean and non-Euclidean with History, 3rd Edition*. Pearson Prentice Hall, 2005.

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[7] Mastin, Luke. "Indian Mathematics." The Story of Mathematics, 2010. <http://www.storyofmathematics.com/indian.html>.

[8] Nelson, David, and Joseph, George G., and Williams, Julian. *Multicultural Mathematics: Teaching Mathematics from a Global Perspective*. Oxford University Press, 1993.

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