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ADVANCED ENRICHMENT TOPICS IN AN HONORS GEOMETRY COURSE

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The essay of Kayla M. Woods is hereby accepted:

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Advisor – Douglas A. Norris

I certify that this is the original document

____ Date_____________

Author – Kayla M. Woods

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Introduction

"You're a math teacher! What course do you teach?" "Geometry!" "Oh, I hated geometry. I was good at algebra. They say if you're bad at one, you're good at the other." If I had a dollar for every similar interaction…

The general population misses out on the beauty of mathematics and all it has to offer because they limit its potential to that which they are familiar: number-crunching and processes. While visual and spatial reasoning skills play a large role in students' struggles, the bigger issue is that the focus of geometry – as in other upper level mathematics courses – shifts from familiar arithmetic and solving equations to constructing logical arguments with precision.

One of the biggest obstacles I have faced teaching Honors Geometry is making sure my students are challenged at the appropriate level compared to our other geometry courses. What does a more rigorous course mean? More proofs? Tedious, ad nauseam detail? Eye-crossing diagrams? More complex algebra? While all of these ideas can be integrated into honors curriculum, my goal has been this: expose students to the bigger picture of mathematics. I have classes of students who are ready and waiting to have their minds expanded. And while expectations and depth of content are at a top level in the curriculum, there is still a need for these excelling students to have the opportunity to expand what they know about mathematics as a whole.

In response to "what does a more rigorous class mean?", the van Hiele Levels of Geometric Understanding offer a solution. The levels and indicators of each stage are as follows:

1. *Visualization* – Students can identify shapes at a basic level; but may not see an obtuse scalene triangle as a triangle because it looks different from an equilateral triangle.
2. *Analysis* – Students can compare and contrast shapes; but cannot say what properties are necessary and sufficient to describe an object.

3. *Abstraction* – Students can compose meaningful definitions and construct informal arguments, as in concluding squares are a type of rectangle.
4. *Deduction* – Students can construct proofs and determine necessary and sufficient conditions; conventional high school geometry proofs are achievable at this level.
5. *Rigor* – Students can establish and compare mathematical systems, understand non-Euclidean geometries, and utilize various proof methods [4].

As the course progresses, students travel through these levels of understanding. Ideally, Honors Geometry students end the year with a firm understanding at the depth of level 4. While my students are not fully ready for immersion into level 5 content, I do believe they are capable of comparing mathematical systems and structures to begin to experience the broader scope of mathematics. In providing them with these opportunities, the rigor and appropriate challenge level for Honors Geometry students can be better achieved.

The following enrichment opportunities were developed from content covered in graduate courses at John Carroll University. Each lesson is associated with a high school geometry topic, but proceeds to explore topics in upper-level mathematics. In Lesson 1, students will use rigid transformations that map a figure onto itself to explore the algebraic structure of a group. In Lesson 2, the focus is on understanding the structure of Euclidean geometry in order to contrast it to the non-Euclidean geometry of the sphere. Lesson 3 shifts the geometric focus of "shape and size" to "shape, but not size" to learn the basics of topology. Continuing in the realm of topology, Lesson 4 has students analyze geometric objects using counting techniques to determine if objects are topologically equivalent. Lastly, Lesson 5 acknowledges infinities in geometry – lines extend infinitely, there are an infinite number of points on a line, etc. – and seeks to focus on the concept of infinity itself. As these lessons require students to utilize knowledge of the topics covered in their geometry course, it is intended for these lessons to be used near the end of the year so that students are at the appropriate level of understanding.

While my primary goal is to expose my students to the broader scope of mathematics, my secondary goal is to help other teachers do the same. With that goal in mind, each lesson has a descriptive plan, student document, and teacher answer document. For lessons that require it, I've also created supplementary resources pages. My honors geometry course is aligned to the Common Core Standards. But since these lessons extend the content beyond the scope of Euclidean geometry, the lessons will be aligned to a subset of the seven Common Core Mathematical Practices (MP).

MP.1 – Make sense of problems and persevere in solving them.

MP.2 – Reason abstractly and quantitatively.

MP.3 – Construct viable arguments and critique the reasoning of others.

MP.4 – Model with mathematics.

MP.5 – Use appropriate tools strategically.

MP.6 – Attend to precision.

MP.7 – Look for and make use of structure.

My classroom follows the Core Connections Geometry curriculum by College Preparatory Mathematics (CPM). As such, my students are trained and comfortable working with discovery and team-based learning. My students are typically in groups of four, with the occasional group of three students. Unless otherwise stated, it should be assumed that students are in groups of four. I've provided background information on what I expect students to know prior to each lesson and the overarching goal for the day. Each lesson begins with an introduction that transitions students from prior knowledge into the mathematical ideas that are beyond the curriculum. The "Suggested Lesson Activity" provides tips for teachers, questions to ask groups as the teacher circulates the room, discussion points to listen for, suggestions for modifications, and general pacing for the lesson. Problems referenced in the "Suggested Lesson Activity" will be found in the student and teacher documents that correspond to the lesson. There will be some questions that are best completed as a teacher-led, whole-class discussion. But the formatting of the student handouts should allow for group autonomy. There will be times to pull the class together as a whole to verify accurate conclusions, but most checking is done as the teacher circulates the room. Students should check with the teacher as they complete problems. It is important that the teacher plays a roll of coach and guide rather than lecturer; teachers should probe students' thinking with questions rather than telling them the correct answer. Most importantly, the teacher should ask how students came to their conclusions, as justification and logical arguments are a main focus in geometry courses.

My plan is to utilize these lessons, or a subset of them, on the school days leading up to final exams. As such, my intention is to give exam review problems for homework. However, since other teachers may have more time for students to try these topics independently, I've provided suggestions for homework assignments. The lessons do not have to be completed in any particular order, except the lesson on topology; it should be completed before the lesson investigating Euler numbers.

Lesson 1: Exploring Groups

Background Knowledge & Overview:

- ***Prior Knowledge:*** Up to this point in their mathematical careers my students have worked with transformations in the following ways:
  - Informally, calling them flips, turns, and slides. The basic concepts were covered in courses prior to geometry, and the motions were typically not done on the coordinate plane.
  - Formally, referring to them as rigid transformations and dilations. Figures were analyzed both on and off of the coordinate plane. The analysis extended further to determine triangle similarity or congruence.
  - Algebraically on the coordinate plane. Students studied transformations of linear functions and quadratics in Algebra 1.
- ***Transition to New Ideas:*** In each of the ways described above, there are an infinite number of transformations that can be performed on a figure. For example, a quadrilateral can be rotated by any degree about any point. Parabolas can be translated in any direction, by any distance. But if only the transformations that carry a figure on to itself are considered, there are a finite number of transformations that can be performed on a figure.
- ***Lesson Overview:*** With this finite number of transformations, students will be able to perform operations with transformations. The table of results will then be analyzed to discover the mathematical structure known as a group.

Goal:

- This lesson utilizes students' knowledge of rigid transformations to learn about properties of groups. Groups are discussed in depth in upper-level mathematics courses at the college level. Groups are used in modular arithmetic, molecular structures (chemistry), cryptography, and are used in the development of graph theory and coding (video games).

Objective:

- To develop the definition of a group using operations of rigid transformations.

Standards:

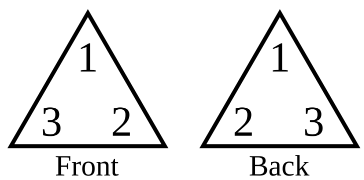
- MP.6 – Attend to precision
- MP.7 – Look for and make use of structure.

Length of Activity:

- One class period (approximately 50 minutes)

Materials:

- Equilateral Triangles
    - Option 1: Each student receives a cut-out equilateral triangle. Whether hand-written or printed, students should have the vertices labeled 1, 2, 3 on the front and the back of the cut-out as demonstrated below.



    - Option 2: Students could use tracing paper and trace the equilateral triangle from Problem 1 on the Student Handout.
- Student Handout: Exploring Groups
- Teacher Answer Document: Exploring Groups

Suggested Lesson Activity:

Groups of four students work well for the format of this lesson.

Introduce the lesson by reading the objective and then have students turn to a partner for 15 seconds to say everything they remember about rigid transformations. Then discuss the "Transition to new idea" section from above. It is important to ensure that students understand the concept "a figure is carried on to itself."

Instruct students to complete Problem 1. Have students work with their teams for approximately 5 minutes to come up with the single rigid transformations that, after being performed, carry the equilateral triangle onto itself. Note: If students struggle with the prompt, an alternative is: "How many different ways can you rearrange the vertex labels (1, 2, 3)? List all of the possibilities. Then go back and state which rigid transformation yields each result." It is important to ensure that students realize that rotating 120° clockwise about the center of the triangle is the same as rotating 240° counter-clockwise about the center of the triangle. Similarly, it will be helpful to take time to discuss that a 360° rotation has the same resulting configuration as a 0° rotation.

As a class, create a list of all of the possibilities. R_1 describes a rotation that is a 120° rotation clockwise about the center of the figure. R_2 describes rotating the triangle 120° twice (240°) about the origin. R_3 represents three 120° rotations (360°); later in Problem 5 this will be referred to as the identity I since it has the same configuration as the 0° rotation. Labels F_T, F_R, and F_L denote reflections across the lines of symmetry through the top, right, and left vertices, respectively. Ensure all students are labeling the transformations in the same way. For example, every student has the same R_3 or identity triangle: vertex 1 at the top, vertex 2 on the right, and vertex 3 on the left.

Have students complete Problems 2 and 3 with their groups. Circulate to assess student discussion and progress and have teams check their answers as they go.

Before doing Problem 3, it is helpful explain to students that they will be organizing their information in a chart much like a multiplication table. Drawing a multiplication table of the numbers 1 through 5 and filling in some cells will be a helpful reference later when describing the identity, associativity, commutativity, and closure of a group. When making the comparisons, note that the 5-by-5 multiplication table itself is not a group, because closure is not met. For example, $(4)(5) = 20,$ but 20 is not in the set of $\{1, 2, 3, 4, 5\}$.

Problem 4 requires students to perform two rigid transformations and determine the single rigid transformation with the same resulting triangle. As this component of the lesson could be time consuming, assigning quadrants of the table to different groups of students will save more class time for the discussion of mathematical groups.

Problem 5 is intended to be conducted as a teacher-led discussion so that students can fill in the appropriate vocabulary. After completing Problem 5, direct students to complete Problems 6 and 7 on the student handout.

Closure:

Use Problem 8 as the closure problem for the day. The main goal of this problem is to show that two groups of the same size do not necessarily have the same structure.

Suggested Homework:

Have students complete the group table of symmetries of a rectangle and ask students a few clock arithmetic questions.

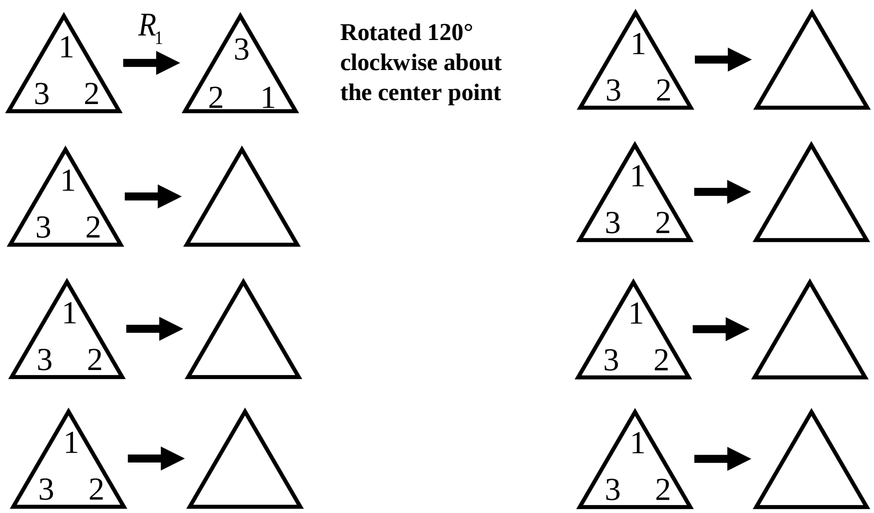
Exploring Groups Name:________________________

Student Handout

Now that we've discussed rigid transformations, we can use them to study other topics in mathematics.

RIGID TRANSFORMATIONS WITH EQUILATERAL TRIANGLES

1. Sydney and her team were discussing all the different rigid transformations that could be performed on an equilateral triangle. They noticed that if they reflected or rotated in specific ways, the triangle was "carried onto itself." For example, when Sydney rotated her cut-out equilateral triangle 120° clockwise about the center point, the rigid transformation completely overlapped the original triangle. Sydney wondered how many distinctly different ways the triangle could be transformed so that it is carried onto itself. Sydney guessed that there were five, Katie thought there were more. Who is correct? As you experiment, note the position of the resulting vertices and describe the type of rigid transformation you used. Sydney's is filled in for you.



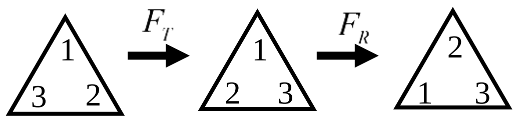
2. With your teacher and the class, come up with a label for each type of motion above. Then, write the list of motions in the brackets below, using commas to separate them.

{____________________}

The set above is known as the **Symmetries of an Equilateral Triangle**.

3. Molly wondered what would happen if two rigid transformations are performed, one right after the other. "Will we always get a member of the Symmetries of an Equilateral Triangle set?

   a. Dillon says, "Here's what happened when I reflected over vertex 1 and then reflected over the position that was originally vertex 2."



   What single rigid transformation yields the same result? ____________.

   b. To answer Molly's question, determine the single rigid transformation that has the outcome of performing two rigid transformations, one followed by another. Write the resultant transformation in the corresponding spot in the table below. If the answer is not one of the options from Problem 2, create a new label. Dillon's answer has already been recorded for you.

		2nd Rigid Transformation					
		R_3	R_1	R_2	F_T	F_R	F_L
1st Rigid Transformation	R_3						
	R_1						
	R_2						
	F_T					R_2	
	F_R						
	F_L						

4. Analyzing the table.

   a. Were there any new transformations? ______ If so, how did you name them?

b. What do you notice about each row in the table? Is the same true in each column?

c. Were there any sections of the table that were easier to complete? Justify.

INTRO TO GROUP THEORY – to be completed with your teacher

5. The symmetries of an equilateral triangle can be classified as a **group** because
   - it is a set: $\{R_3, R_1, R_2, F_T, F_R, F_L\}$
   - with a binary operation: performing two rigid transformations, one followed by another

   and it has the following properties:

   a. ___________________ - when the operation is performed, the result is an element in the set. (i.e. we didn't get a "new" result.)

   b. ___________________ - this is the same as the property used in addition:

   Ex: $(3+2)+4=3+(2+4)$

   Try it! Is $R_1 \to (R_2 \to F_T)$ the same result as $(R_1 \to R_2) \to F_T$? Justify.

   c. ___________________ - This is the same concept as adding zero to any number; a number added to zero is itself. Which motion in our table is the identity? ___________ This element in the set can be written as "I."

   d. ___________________ - when performing the operation between an element and its inverse, the result is the identity. Every element in the set must have an inverse. Find the inverses of the following:

   $R_3 \to$ _____ $= I$ $R_1 \to$ _____ $= I$ $R_2 \to$ _____ $= I$

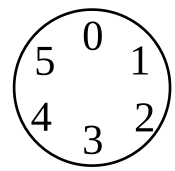
   $F_T \to$ _____ $= I$ $F_R \to$ _____ $= I$ $F_L \to$ _____ $= I$

MORE GROUPS!

6. Without knowing it, you've been utilizing groups your whole life! Clock arithmetic or modular arithmetic is another group that is frequently used. Consider the following: if it is 11 o'clock in the morning and school ends in three hours, do you say that school releases at 14 o'clock? No! What time does school get out? ______. How did you determine your answer?

7. When you're calculating time on a clock, as in Problem 6, the **modulus** is 12. Let's look at a simpler clock of modulo 6.
There are 6 elements in the set: $\{0,1,2,3,4,5\}$. This is referred to as $\mathbf{Z}_6$, the set of non-negative integers less than 6.

   a. When working with clock addition in modulo 6, the result of $1+2$ is still 3. But when we add $3+4$, it no longer equals 7 because seven is not in the original set. (The group would not have closure!) Use the clock at right. Start at 3 and then travel 4 spots in the clockwise direction. Where did you land? ________ So $(3+4)\bmod 5 =$ ___.



   b. Use the clock from part (a) to complete the table. The top left corner of the table tells us the operation is addition.

+		2^{nd} Number					
		0	1	2	3	4	5
1^{st} Number	0						
	1						
	2						
	3						
	4						
	5						

   c. Use the results from the table and the questions below to verify that $\mathbf{Z}_6$ is a group.

i. Closure: were there any “new” results in the table?

ii. Associative: does $(2+4)+4$ have the same result as $2+(4+4)$? Show work to support your answer.

iii. Identity: does this group have an identity? If so, what is it?

iv. Inverses: State the inverses for each element in $\mathbf{Z}_6$.

d. There are patterns in this table that makes it aesthetically pleasing and simplifies the process of filling in all of the boxes. What do you notice?

e. Did you notice the symmetry along the diagonal of the table? That means this group has an additional characteristic referred to as the **commutative** property. Show an example of this property below:

f. It can be shown that $\mathbf{Z}_6$ is commutative and the group of the Symmetries of the Equilateral Triangle is not. For example, show $R_1 + F_1 \neq F_1 + R_1$.

CLOSURE

8. Take a look back at the completed tables for the Symmetries of a Triangle and $\mathbf{Z}_6$.
   a. What do you notice about the size of each group?

   b. What do you notice about the symmetry along the diagonal in the two groups? What property relates to this characteristic?

   c. Is the structure of both groups the same?

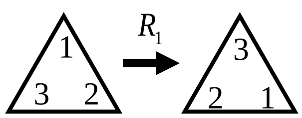
Exploring Groups Name:_______________________

Teacher Answer Document

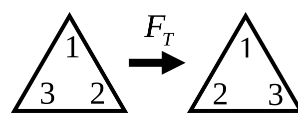
Now that we've discussed rigid transformations, we can use them to study other topics in mathematics.

RIGID TRANSFORMATIONS WITH EQUILATERAL TRIANGLES

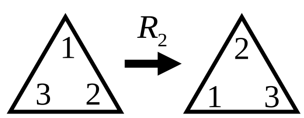
1. Sydney and her team were discussing all the different rigid transformations that could be performed on an equilateral triangle. They noticed that if they reflected or rotated in specific ways, the triangle was "carried onto itself." For example, when Sydney rotated her cut-out equilateral triangle 120° clockwise about the center point, the rigid transformation completely overlapped the original triangle. Sydney wondered how many distinctly different ways the triangle could be transformed so that it is carried onto itself. Sydney guessed that there were five, Katie thought there were more. Who is correct? As you experiment, note the position of the resulting vertices and describe the type of rigid transformation you used. Sydney's is filled in for you.



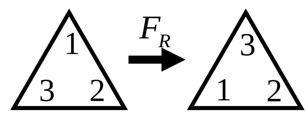
Rotated 120° clockwise about the center point



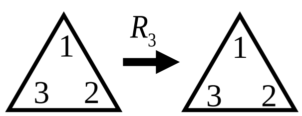
Reflect over the line of symmetry the top vertex



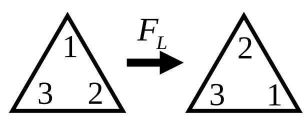
Rotated 240° clockwise about the center point



Reflect over the line of symmetry at the right vertex



Rotated 360° clockwise about the center point



Reflect over the line of symmetry at the left vertex

Katie is correct; there are 6 rigid transformations that carry the figure onto itself in distinctly different ways.

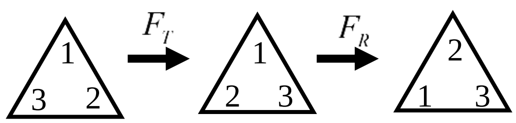
2. With your teacher and the class, come up with a label for each type of motion above. Then, write the list of motions in the brackets below, using commas to separate them.

$$\{ R_3, R_1, R_2, F_T, F_R, F_L \}$$

The set above is known as the **Symmetries of an Equilateral Triangle**.

3. Molly wondered what would happen if two rigid transformations are performed, one right after the other. “Will we always get a member of the Symmetries of an Equilateral Triangle set?”

   a. Dillon says, “Here’s what happened when I reflected over vertex 1 and then reflected over the position that was originally vertex 2.”



   What single rigid transformation yields the same result? R_2

   b. To answer Molly’s question, determine the single rigid transformation that has the outcome of performing two rigid transformations, one followed by another. Write the resultant transformation in the corresponding spot in the table below. If the answer is not one of the options from Problem 2, create a new label. Dillon’s answer has already been recorded for you.

		2nd Rigid Transformation					
		R_3	R_1	R_2	F_T	F_R	F_L
1st Rigid Transformation	R_3	R_3	R_1	R_2	F_T	F_R	F_L
	R_1	R_1	R_2	R_3	F_R	F_L	F_T
	R_2	R_2	R_3	R_1	F_L	F_T	F_R
	F_T	F_T	F_L	F_R	R_3	R_2	R_1
	F_R	F_R	F_T	F_L	R_1	R_3	R_2
	F_L	F_L	F_R	F_T	R_2	R_1	R_3

4. Analyzing the table.
    a. Were there any new transformations? No. If so, how did you name them? N/A
    b. What do you notice about each row in the table? Is the same true in each column? Each row and each column has every rigid transformation exactly once.
    c. Were there any sections of the table that were easier to complete? Justify. There are 4 quadrants in the table; two quadrants have only rotations, the other two quadrants have only reflections. A rigid transformation with R_3 ended up being itself.

INTRO TO GROUP THEORY – to be completed with your teacher

5. The symmetries of an equilateral triangle can be classified as a **group** because
    - it is a set: $\{R_3, R_1, R_2, F_T, F_R, F_L\}$
    - with a binary operation: performing two rigid transformations, one followed by another

    and it has the following properties:

    a. Closure - when the operation is performed, the result is an element in the set. (i.e. we didn't get a "new" result.)

    b. Associative - this is the same as the property used in addition:

    Ex: $(3+2)+4=3+(2+4)$

    Try it! Is $R_1 \to (R_2 \to F_T)$ the same result as $(R_1 \to R_2) \to F_T$? Justify.

    $R_1 \to F_L = F_T$ $\qquad$ $R_3 \to F_T = F_T$

    The series of transformations both yield F_T, so the way the rigid transformations are grouped does not change the outcome.

    c. Identity - This is the same concept as adding zero to any number; a number added to zero is itself. Which motion in our table is the identity? R_3 This element in the set can be written as "*I*."

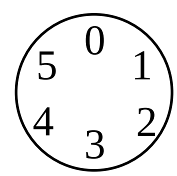
d. Inverse - when performing the operation between an element and its inverse, the result is the identity. Every element in the set must have an inverse. Find the inverses of the following:

$R_3 \rightarrow R_3 = I$	$R_1 \rightarrow R_2 = I$	$R_2 \rightarrow R_1 = I$
$F_T \rightarrow F_T = I$	$F_R \rightarrow F_R = I$	$F_L \rightarrow F_L = I$

MORE GROUPS!

6. Without knowing it, you've been utilizing groups your whole life! Clock arithmetic or modular arithmetic is another group that is frequently used. Consider the following: if it is 11 o'clock in the morning and school ends in three hours, do you say that school releases at 14 o'clock? No! What time does school get out? 2pm. How did you determine your answer? Answers vary: $11+3=14$, but then subtract 12 to determine the time; after you count to 12 o'clock, start over again at 1, etc.
7. When you're calculating time on a clock, as in Problem 6, the **modulus** is 12. Let's look at a simpler clock of modulo 6.
   There are 6 elements in the set: $\{0,1,2,3,4,5\}$. This is referred to as $\mathbf{Z}_6$, the set of non-negative integers less than 6.

   a. When working with clock addition in modulo 6, the result of $1+2$ is still 3. But when we add $3+4$, it no longer equals 7 because seven is not in the original set. (The group would not have closure!) Use the clock at right. Start at 3 and then travel 4 spots in the clockwise direction. Where did you land? ____1____ So $(3+4)\bmod 5 =$ __1__.



b. Use the clock from part (a) to complete the table. The top left corner of the table tells us the operation is addition.

+		2^{nd} Number					
		0	1	2	3	4	5
1^{st} Number	0	0	1	2	3	4	5
	1	1	2	3	4	5	0
	2	2	3	4	5	0	1
	3	3	4	5	0	1	2
	4	4	5	0	1	2	3
	5	5	0	1	2	3	4

c. Use the results from the table and the questions below to verify that $\mathbf{Z}_6$ is a group.

i. Closure: were there any "new" results in the table? No.

ii. Associative: does $(2+4)+4$ have the same result as $2+(4+4)$? Show work to support your answer.

$(2+4)+4$ $\quad$ $2+(4+4)$

$=(0)+4$ $\quad$ and $\quad$ $=2+(2)$ $\quad$ so the results are the same.

$=4$ $\quad$ $=4$

iii. Identity: does this group have an identity? If so, what is it? Yes; 0.

iv. Inverses: State the inverses for each element in $\mathbf{Z}_6$.

$0+0=0$	$3+3=0$
$1+5=0$	$4+2=0$
$2+4=0$	$5+1=0$

0 and 3 are inverses of themselves. 1 and 5 are inverses of each other, and 2 and 4 are inverses of each other.

d. There are patterns in this table that makes it aesthetically pleasing and simplifies the process of filling in all of the boxes. What do you notice? Each diagonal (moving up and to the right) is composed of the same number. Along the main diagonal (going down and to the right from the top , left cell), there is symmetry.

e. Did you notice the symmetry along the diagonal of the table? That means this group has an additional characteristic referred to as the **commutative** property. Show an example of this property below:

$2+1=3$ and $1+2=3$

f. It can be shown that $\mathbf{Z}_6$ is commutative and the group of the Symmetries of the Equilateral Triangle is not. For example, show $R_1 + F_T \neq F_T + R_1$. $R_1 + F_T = F_R$ and $F_T + R_1 = F_L$, so $F_R \neq F_L$.

CLOSURE

8. Take a look back at the completed tables for the Symmetries of a Triangle and $\mathbf{Z}_6$.
   a. What do you notice about the size of each group? They both have 6 elements.
   b. What do you notice about the symmetry along the diagonal in the two groups? What property relates to this characteristic? There is not symmetry along the main diagonal in the Symmetries of a Triangle table, but there is for $\mathbf{Z}_6$. So the group of Symmetries of a Triangle is not commutative but the group of $\mathbf{Z}_6$ is commutative.
   c. Is the structure of both groups the same? No. So we can have two sets of the same size, but they both behave differently.

Lesson 2: Exploring Non-Euclidean Geometry

Background Knowledge & Overview:

- ***Prior Knowledge:***
    - High school geometry courses examine the structure of Euclidean geometry. Students are typically unaware of this because the common notions and axioms upon which the course is built are not particularly emphasized. So students will need to recognize the structure they have been using in order to understand the difference between Euclidean and non-Euclidean geometry.
    - There are many references to theorems that have been discussed throughout the high school geometry course.
    - For calculation purposes, students need to know the formula for the area of a triangle, the triangle angle sum theorem, the surface area formula for a sphere. Students should also have a familiarity with a globe.
- ***Transition to New Ideas:*** If we consider changing Euclid's postulates, most notably the 5^{th} postulate, we can explore the geometry that occurs on a sphere.
- ***Lesson Overview:*** Students will analyze Euclid's first five postulates and use lines of longitude and the equator on a globe to work with areas of triangles in spherical geometry. Students will then compare and contrast Euclidean and spherical geometry.

Goal:

- This lesson utilizes students' knowledge of their high school geometry course to learn about other branches of geometry. Non-Euclidean geometries are another important component of upper-level mathematics courses at the college level. Acknowledging the existence of non-Euclidean geometry will help students realize that there is more to mathematics than the courses they will take in high school.

Objective:

- To compare and contrast Euclidean and non-Euclidean geometries.
- To solve basic problems involving triangles in spherical geometry.

Standards:

- MP.1 – Reason abstractly and quantitatively.
- MP.5 – Use appropriate tools strategically.
- MP.7 – Look for and make use of structure.

Length of Activity:

- One class period (approximately 50 minutes)

Materials:

- A Globe
- Student Handout: Exploring Non-Euclidean Geometry
- Teacher Answer Document: Exploring Non-Euclidean Geometry

Suggested Lesson Activity:

The recommended format for this lesson is to give groups a few minutes to work on a problem or two and then discuss the results and ramification with the whole. This will keep the lesson moving.

Have a student volunteer read the introduction. Then give groups about five minutes to complete Problems 1, 2, and 3. Then have a class discussion about the results.

Groups should then be given approximately three minutes to complete Problems 4 and 5. After discussing the answers together, talk through Problems 5 and 6 in a whole-class discussion. These problems set up the transition into spherical geometry.

Problem 7 has groups brainstorm everything they remember about spheres. Referring them to a globe will help students shift their thinking to objects drawn on the surface of a sphere. This discussion should not take long. The questions that follow are really where teachers should focus. After about a minute, have each group share one fact. It's important to define a line as a great circle for the next question.

Have the class read Question 8 together and then give students a minute to think alone, a minute to discuss with their groups, and then briefly discuss the question together to ensure that students understand why there are no parallel lines in spherical geometry.

After completing Problem 9, make sure to emphasize that the sum of the angles in a spherical triangle being larger than 180° is related to changing Euclid's fifth postulate. In Euclidean geometry, a triangle can never have two 90° angles because the lines would be parallel.

Students will then be led by prompts to investigate if the Euclidean formula for the area of a triangle can be used to calculate the area of a triangle on a sphere. Making this comparison will rely on students' knowledge of finding the surface area of a sphere. If time is running short this could be a teacher-led discussion. Part (f) of Problem 10 extends the lesson and addresses the intuitive question "Is there a formula for the area of a spherical triangle?" While time may not allow for this part of Question 10, it could be assigned as homework. Students may find it intriguing that the area formula uses angle measures instead of side lengths.

Closure:

- At the end of the lesson have students tell a partner two things they learned, one thing they are confused about, and one thing they are curious about based on today's lesson. Take time to discuss with the class. Some responses could lead nicely into the homework suggestion.

Homework Suggestion:

- After completing Problem 9, students may naturally wonder how large the sum of angles in a triangle can be in spherical geometry. Researching this topic could be a homework investigation as an extension to the lesson.

Exploring Non-Euclidean Geometry Name:________________________

Student Handout

Did you know there are other types of Geometries than the one we've studied all year? It's true! In this course, we've been learning Euclidean geometry. The entire course depends on five axioms. If we change some of the axioms, we can explore the basics of the non-Euclidean geometry called spherical geometry.

EUCLIDEAN GEOMETRY

1. Euclidean geometry gets its name from the Greek mathematician Euclid, who lived approximately 2300 years ago. He wrote the first mathematical textbook called "The Elements." In it, he provided the foundation from which he built plane (2D) and solid (3D) geometry.

   Draw pictures to represent each of Euclid's Axioms (also called Euclid's Postulates) below.

   I. Through any 2 points there is exactly 1 line.

   II. Line segments can be extended.

   III. Circles exist through any point with any given radius.

   IV. All right angles are congruent.

   V. If the sum of Same Side Interior angles is less than 180°, then the two lines intersect on that side.

2. Axiom V is equivalent to the statement: Given a line *m* and a point *P* not on *m*, there exists exactly one line *n* parallel to *m* through the point *P*. Draw a picture that depicts this statement.

3. Using only axioms I to IV, Euclid developed 28 Theorems!
   As you look at the selected list of theorems from these axioms, do you notice any well-known theorems missing?

   - If two lines intersect, then vertical angles are congruent.
   - If a triangle has two congruent sides, then the angles opposite those sides are also congruent.
   - If two angles form a linear pair, then they are supplementary.
   - In a triangle the side opposite the largest angle is the largest side.
   - The sum of the lengths of any two sides of a triangle is greater than the length of the third.
   - If two lines are cut by a transversal and if the alternate interior angles are equal, then the two lines are parallel.

THE FIFTH POSTULATE

4. Look back at Axiom V in Problem 1. What would happen if the sum of the same side interior angles was equal to 180°?

5. Some of the most frequently used theorems in geometry come from Axiom V. It's also commonly referred to as the Fifth Postulate or the Parallel Postulate. This is the postulate that gives us such powerful tools as the Pythagorean Theorem and the Triangle Angle Sum Theorem. What concepts have we covered this year that used these theorems?

IS IT TRUE?

6. Mathematicians weren't certain that the Parallel Postulate is *always* true. So they asked the question "What happens if we change the structure by eliminating or modifying this postulate?" This is how other geometries were born.

   In Problem 2, it stated that through a point not on a line, there is exactly one parallel line that can be drawn. So if mathematicians negated this statement (i.e. said it was false), then what were they claiming?

SPHERICAL GEOMETRY

7. Let’s consider a geometry that occurs only on the surface of a sphere. Brainstorm with your team to make a list of facts we already know about spheres. It may be helpful to think of a globe.

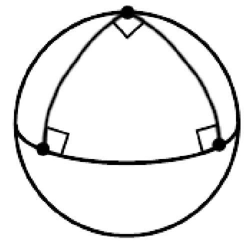
8. Aaron and Jared are examining the following definitions in spherical geometry:
    - Lines are defined as great circles.
    - Edges are arcs of great circles.

   Aaron thinks that the latitudinal markings on the globe are parallel because they do not intersect. Jared doesn’t think that parallel lines exist in spherical geometry. Who is correct? Explain your reasoning. Use a model of a globe or a sphere to help visualize as you analyze.

TRIANGLES ON A SPHERE

9. Aaron wondered what else is different on a sphere. He noticed that the lines of longitude on a globe made various sizes of triangles with the equator. After observing for a few moments, he exclaimed, “Jared, look! There is a triangle with three right angles!” Is Aaron correct? Use a globe to justify your answer.

10. “Does this mean that the formula for the area of a triangle isn’t going to work?” Jared asked. “Everything is changing!” Find the area of Jared and Aaron’s triangle from Problem 9 by completing the steps below.



a. Make a prediction. Do you think that the Euclidean geometry formula for the area of a triangle will accurately calculate the area of a triangle on the sphere?

b. The sphere at right has a radius of 2 units. Find the arc length for the edge of the triangle that is along the equator. Keep in mind that the arc measure for this edge is 90°.

c. The height of the triangle is also an arc with a 90° angle measure, so the triangle’s height is the same length as the base. Use the information to calculate the area of a triangle according to Euclidean geometry.

d. To see if this answer is correct, find the area of the triangle by calculating in terms of the fraction of the sphere’s surface area.

e. Are the results the same? What does this say about the Euclidean formula for the area of a triangle in spherical geometry.

f. Another way to find the area of the spherical triangle is broken down into steps below.

   i. 90° is $\frac{\pi}{2}$ in radian measure. Find the sum of the interior angles of the triangle in radian form. Then subtract π.

   ii. The radius of the sphere is 2. Square it, then multiply it by the answer from part (i).

   iii. Conjecture a spherical triangle area formula that uses angles instead of side lengths.

Exploring Non-Euclidean Geometry Name:________________________

Teacher Answer Document

Did you know there are other types of geometries than the one we've studied all year? It's true! In this course, we've been learning Euclidean geometry. The entire course depends on five axioms. If we change some of the axioms, we can explore the basics of the non-Euclidean geometry called spherical geometry.

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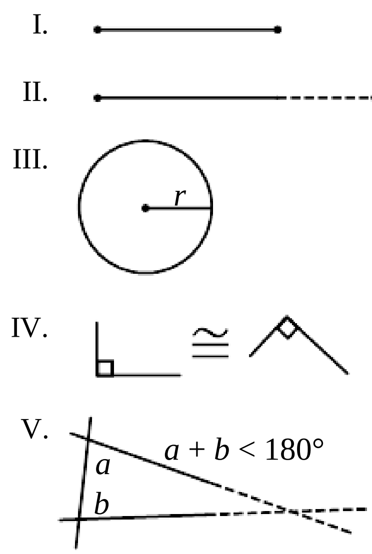
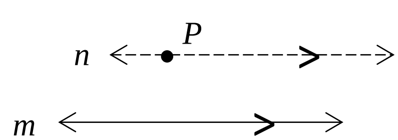
   III. Circles exist through any point with any given radius.

   IV. All right angles are congruent.

   V. If the sum of Same Side Interior angles is less than 180°, then the two lines intersect on that side.



2. Axiom V is equivalent to the statement: Given a line *m* and a point *P* not on *m*, there exists exactly one line *n* parallel to *m* through the point *P*. Draw a picture that depicts this statement.



3. Using only axioms I to IV, Euclid developed 28 Theorems!
   As you look at the selected list of theorems from these axioms, do you notice any well-known theorems missing? Pythagorean Theorem, Triangle Angle Sum Theorem

   - If two lines intersect, then vertical angles are congruent.
   - If a triangle has two congruent sides, then the angles opposite those sides are also congruent.
   - If two angles form a linear pair, then they are supplementary.
   - In a triangle the side opposite the largest angle is the largest side.
   - The sum of the lengths of any two sides of a triangle is greater than the length of the third.
   - If two lines are cut by a transversal and if the alternate interior angles are equal, then the two lines are parallel.

THE FIFTH POSTULATE

4. Look back at Axiom V in Problem 1. What would happen if the sum of the same side interior angles was equal to 180°? If same-side interior angles are supplementary, then lines cut by a transversal are parallel.

5. Some of the most frequently used theorems in geometry come from Axiom V. It's also commonly referred to as the Fifth Postulate or the Parallel Postulate. This is the postulate that gives us such powerful tools as the Pythagorean Theorem and the Triangle Angle Sum Theorem. What concepts have we covered this year that used these theorems? Angles in polygons, trigonometry, properties of quadrilaterals, etc.

IS IT TRUE?

6. Mathematicians weren't certain that the Parallel Postulate is *always* true. So they asked the question "What happens if we change the structure by eliminating or modifying this postulate?" This is how other geometries were born.

   In Problem 2, it stated that through a point not on a line, there is exactly one parallel line that can be drawn. So if mathematicians negated this statement (i.e. said it was false), then what were they claiming? The claim is that there can either be more than one parallel line through the point not on the given line or that there are no parallel lines.

SPHERICAL GEOMETRY

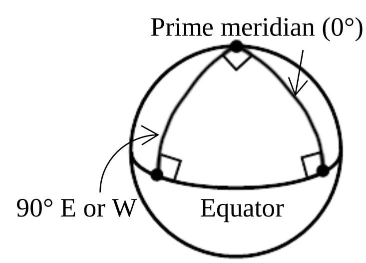
7. Let's consider a geometry that occurs only on the surface of a sphere. Brainstorm with your team to make a list of facts we already know about spheres. It may be helpful to think of a globe. Great circles have the same diameter as the sphere, lines of longitude are great circles, the equator is a great circle, the surface area is $4\pi r^2$, half of a sphere is called a hemisphere, etc.

8. Aaron and Jared are examining the following definitions in spherical geometry:
   - Lines are defined as great circles.
   - Edges are arcs of great circles.

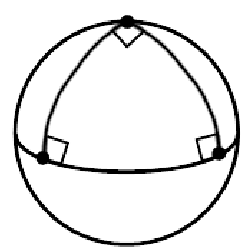
   Aaron thinks that the latitudinal markings on the globe are parallel because they do not intersect. Jared doesn't think that parallel lines exist in spherical geometry. Who is correct? Explain your reasoning. Use a model of a globe or a styrofoam sphere to help visualize as you analyze. There are no parallel lines in spherical geometry. "Lines" of latitude are not great circles, so they are not considered lines in spherical geometry.

TRIANGLES ON A SPHERE

9. Aaron wondered what else is different on a sphere. He noticed that the lines of longitude on a globe made various sizes of triangles with the equator. After observing for a few moments, he exclaimed, "Jared, look! There is a triangle with three right angles!" Is Aaron correct? Use a globe to justify your answer. Yes, Aaron is correct. Using the equator, the prime meridian, and the longitudinal line at 90° W (or 90° E), the intersections of each pair of lines is a right angle.



10. "Does this mean that the formula for the area of a triangle isn't going to work?" Jared asked. "Everything is changing!" Find the area of Jared and Aaron's triangle from Problem 9 by completing the steps below.



    a. Make a prediction. Do you think that the Euclidean geometry formula for the area of a triangle will accurately calculate the area of a triangle on the sphere? Answers vary.

b. The sphere at right has a radius of 2 units. Find the arc length for the edge of the triangle that is along the equator. Keep in mind that the arc measure for this edge is 90°.

Since the arc measure is 90°, the arc is $\frac{1}{4}$ the circumference of the great circle.

$C = 2\pi r = 2\pi(2) = 4\pi$, so $\frac{1}{4}C = \frac{1}{4}(4\pi) = \pi$.

Therefore the arc measure is π units.

c. The height of the triangle is also an arc with a 90°angle measure, so the triangle's height is the same length as the base. Use the information to calculate the area of a triangle according to Euclidean geometry.

$A = \frac{1}{2}bh = \frac{1}{2}(\pi)(\pi) = \frac{\pi^2}{2} \approx 4.935 \text{ units}^2$.

d. To see if this answer is correct, find the area of the triangle by calculating in terms of the fraction of the sphere's surface area.

$SA = 4\pi r^2 = 4\pi(2)^2 = 16\pi \text{ units}^2$

Since the triangle is $\frac{1}{8}$ of the surface area, the area of the triangle is

$\frac{1}{8}SA = \frac{1}{8}(16\pi) = 2\pi \approx 6.282$ units.

e. Are the results the same? What does this say about the Euclidean formula for the area of a triangle in spherical geometry. No. The formula $A = \frac{1}{2}bh$ does not work in spherical geometry, so there must be another formula.

f. Another way to find the area of the spherical triangle is broken down into steps below.

   i. 90° is $\frac{\pi}{2}$ in radian measure. Find the sum of the interior angles of the triangle in radian form. Then subtract π.

   $\left(\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{2}\right)-\pi=\frac{3\pi}{2}-\pi=\frac{\pi}{2}$

   ii. The radius of the sphere is 2. Square it, then multiply it by the answer from part (i).

   $2^2=4,\ 4\left(\frac{\pi}{2}\right)=2\pi$

   iii. Conjecture a spherical triangle area formula that uses angles instead of side lengths.

   $A=r^2\cdot(\text{sum of the interior angles in the triangle}-\pi)$

   *Note: The angles must be in radians.

Lesson 3: Exploring Topology

Background Knowledge & Overview:

- Note: This lesson is patterned after the discovery-based investigation in the book *Rubber Bands, Baseballs, and Doughnuts: A Book About Topology* by Robert Froman [2].
- ***Prior Knowledge:*** By the end of the school year, students have worked their way through the van Hiele Levels of Geometric Understanding, and consequently are able to classify, compare and contrast, find connections with, and prove properties about figures. Much of what has been accomplished was done through the lenses of similarity and congruence; both of which rely on measurements of angles, sides, or radii of figures.
- ***Transition to New Ideas:*** If we consider classifying figures without the use of measurement, we can explore elements of topology.
- ***Lesson Overview:*** Students will analyze figures topologically and classify them by genus using the visual aid of modeling clay.

Goal:

- Throughout the geometry course, special attention has been given to the shape and size of figures. This lesson shifts students' focus to consider the shape only, with no respect to size, in order to study geometry in the more primitive form of topology.

Objective:

- To classify figures topologically.

Standards:

- MP.2 – Reason abstractly and quantitatively.
- MP.5 – Use appropriate tools strategically.
- MP.7 – Look for and make use of structure.

Length of Activity:

- One class period (approximately 50 minutes)

Materials:

- Spoon with reflective qualities, reflective Christmas bulb ornament, or pictures of people's faces reflected in either item
- Modeling clay
- Plastic knives or unfolded paperclips
- Student Handout: Exploring Topology
- Teacher Answer Document: Exploring Topology

Suggested Lesson Activity:

Have a student volunteer to read the introduction and Problem 1. Then have teams complete the problem. Students should check with the teacher when finished. Then have students do Problems 2 and 3.

After Problem 3, discuss the observations students made and ensure they notice connectivity, consecutive order, and inner and outer regions. Then have a follow up discussion about the invariants of a human face when reflected on a curved, shiny surface like a spoon, Christmas ornament, or the side of a toaster. It can be modeled by a student or discussed using a displayed picture of the reflection. Then have teams complete Problems 4, 5 and 6.

Problems 7-10 are best done as teacher led prompts so that the whole class is working on the same problem simultaneously. Beginning with Problem 7, any mention of "solids" refers to the *surface* of the solids.

After Problem 9, discuss the following as a class:

- How can you slice the surface of the donut so that it ***does*** split into two pieces?
- How can you slice the surface of the donut so that it does ***not*** separate into two pieces?
- What is the genus of a donut?

Closure:

- Problem 12 revisits the alphabet grouping from Problem 1 to see if students can accurately classify a figure by genus.

Homework Suggestion:

- Problems like classifying digits by their genus and questions like Problem 6 – with more specific prompts – and Problem 10.

Exploring Topology Name:________________________

Student Handout

Much of what is accomplished in high school geometry is done through the lenses of similarity and congruence; both of which rely on measurements of angles, sides, or radii of figures. If we consider classifying figures without the use of measurement, we can classify figures according to a different branch of mathematics called topology.

A NEW LOOK AT THE ABC'S

1. Megan, Nick, and Anthony were analyzing the letters ABCDEFG. Each of them found a different way to group the letters based on their characteristics. Work with your team to determine how each student classified the letters. Note: the specific font used for the letters is important.
   a. Megan: {A, E}, {B, C, D, F, G}
   b. Nick: {A}, {B, C, D, E}, {F, G}
   c. Anthony: {A, D}, {B}, {C, E, F, G}

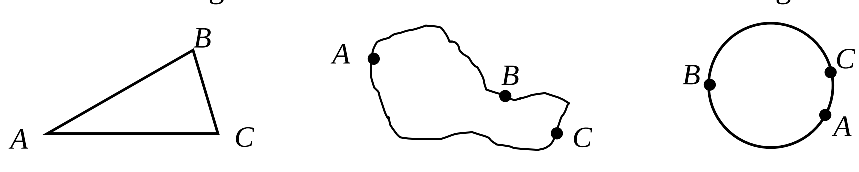
2. Anthony's method was a bit more abstract than Nick's, but both are mathematical. Whether he realized it or not, Anthony classified the letters based on topology. **Topology** is the study of what does not change when line segments and different kinds of shapes are scrunched, stretched, twisted, or distorted in other ways.

   Consider the two descriptions:
   a. The nearest water fountain is downstairs, 33 feet and 7 inches to the left.
   b. The nearest water fountain is downstairs; it's on your left.

   Both descriptions can be helpful, but if you were in a fun-house full of distortion mirrors, which description would still help you locate the water fountain?

INVARIANTS

3. Invariants are attributes that do not change when a figure is scrunched, stretched or distorted. Triangle ABC has been distorted to become the figures to its right.



   List some observations of what is invariant between the two figures

WHAT'S IN A NAME?

4. In topology, instead of having to utilize classifications and names for shapes like right trapezoids, parallelograms, rhombuses, concave polygons, and scalene triangles, we can classify them all with one title: a **simple closed curve**. The triangle, its corresponding blob, and the outline of the reflection of your face in a shiny spoon are all simple closed curves. Draw a few examples of simple closed curves below.

5. Draw an 8 in the margin. The number eight is <u>not</u> a simple closed curve. Why do you think that is?

6. Draw a closed curve that separates the surface to one outside and several insides.

MODEL – *Any mention of the word "solid" is referring to the surface of the solid.*

7. To investigate further, let's go 3D. Consider a basketball; it's like a simple closed curve on a piece of paper because it has exactly one inside and one outside. Is the same true for a snake and a pancake?

8. Get modeling clay from your teacher. Each group should divide and conquer to make two spheres and two snakes per group.
    a. In topology, the figures in Problem 7 are categorized in the same group. That means that it should be possible to change the sphere into a pancake or snake, etc. without cutting or tearing. Discuss how this could be accomplished.

    b. Use an un-bent paperclip or a plastic knife to gently etch a simple closed curve onto the surface of your figure; circles and ovals are recommended. Then slice the figure. (Think cross sections!)

c. All of these figures were sliced into two pieces. In order to keep the figure in one piece, we can make zero slices, so these figures are classified as having a **genus** of 0. Sketch a picture of what the result of your slice looked like.

9. Dino wondered what would happen if he made his modeling clay into a donut.
    a. Does the donut behave in the same way as the pancake? Investigate with your team. Be sure to etch your closed curve before slicing. Note the observations your team makes. What is the genus of a donut? Justify your reasoning.

    b. Can you think of other figures or items with the same genus?

10. A topologist sees no difference between a coffee cup and a donut. Using modeling clay, work with your team to morph your donuts into coffee cups. Remember to change it without cutting or tearing! Check in with your teacher when finished.

MORE THAN DONUTS

11. What types of figures have a higher genus? Brainstorm with your team and sketch a few of your findings.
    a. Figures with a genus of 2.

    b. Figures with a genus of 3, 4, 5, or higher.

CLOSURE

12. Let's revisit Problem 1. Anthony's alphabet grouping was by genus. Work with your team to place the remaining letters in the appropriate genus.

a. Genus 1: {A, D, ____}

b. Genus 2: {B, ____}

c. Genus 0: {C, E, F, G, _______________________________________}

Exploring Topology Name:________________________

Teacher Answer Document

Much of what is accomplished in high school geometry is done through the lenses of similarity and congruence; both of which rely on measurements of angles, sides, or radii of figures. If we consider classifying figures without the use of measurement, we can classify figures according to a different branch of mathematics called topology.

A NEW LOOK AT THE ABC'S

1. Megan, Nick, and Anthony were analyzing the letters ABCDEFG. Each of them found a different way to group the letters based on their characteristics. Work with your team to determine how each student classified the letters. Note: the specific font used for the letters is important.
   a. Megan: {A, E}, {B, C, D, F, G} Vowels, Consonants
   b. Nick: {A}, {B, C, D, E}, {F, G} Vertical line of symmetry, horizontal line of symmetry, no lines of symmetry
   c. Anthony: {A, D}, {B}, {C, E, F, G} 1 loop, 2 loops, no loops

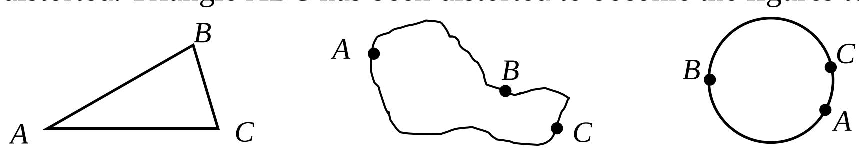
2. Anthony's method was a bit more abstract than Nick's, but both are mathematical. Whether he realized it or not, Anthony classified the letters based on topology. **Topology** is the study of what does not change when line segments and different kinds of shapes are scrunched, stretched, twisted, or distorted in other ways.

   Consider the two descriptions:
   a. The nearest water fountain is downstairs, 33 feet and 7 inches to the left.
   b. The nearest water fountain is downstairs; it's on your left.

   Both descriptions can be helpful, but if you were in a fun-house full of distortion mirrors, which description would still help you locate the water fountain? Description B

INVARIANTS

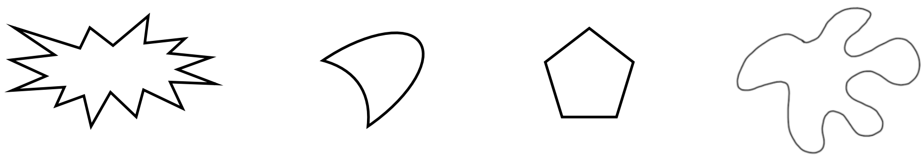
3. Invariants are attributes that do not change when a figure is scrunched, stretched or distorted. Triangle ABC has been distorted to become the figures to its right.



   List some observations of what is invariant between the two figures. *A* still connects to *B* and *C* with one path. Each connection is in the same consecutive order when traveling clockwise around all three figures. There is one inside region and one outside region for all three figures.

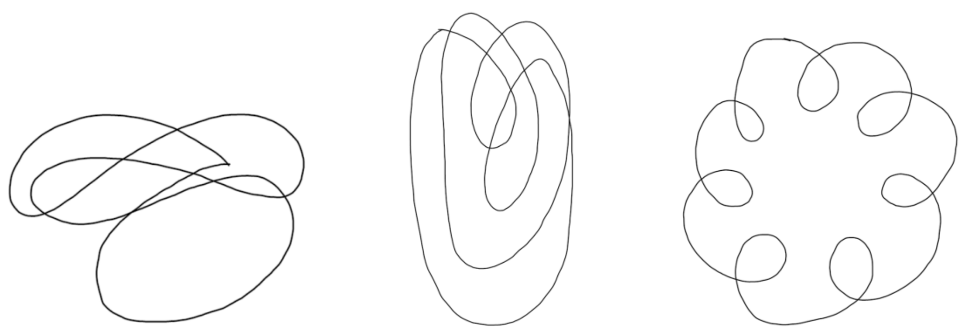
WHAT'S IN A NAME?

4. In topology, instead of having to utilize classifications and names for shapes like right trapezoids, parallelograms, rhombuses, concave polygons, and scalene triangles, we can classify them all with one title: a **simple closed curve**. The triangle, its corresponding blob, and the outline of the reflection of your face in a shiny spoon are all simple closed curves. Draw a few examples of simple closed curves below.



5. Draw an 8 in the margin. The number eight is not a simple closed curve. Why do you think that is? It splits the paper into one outside and two insides, whereas the simple closed curves only split the paper into one outside and one inside region.

6. Draw a closed curve that separates the surface to one outside and several insides. Example answers below.



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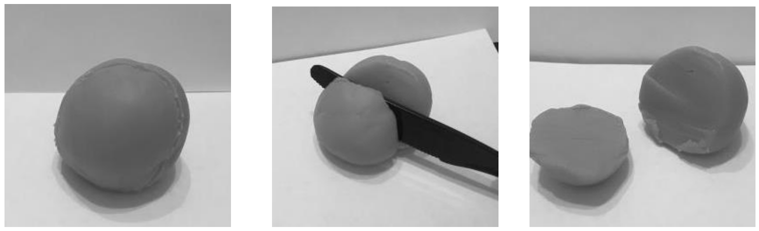
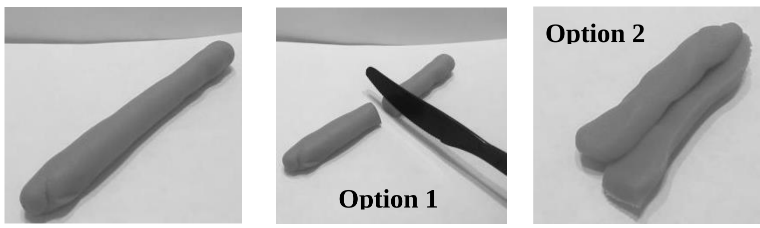
7. To investigate further, let's go 3D. Consider a basketball; it's like a simple closed curve on a piece of paper because it has exactly one inside and one outside. Is the same true for a snake and a pancake? Yes.

8. Get modeling clay from your teacher. Each group should divide and conquer to make two spheres and two snakes per group.
   a. In topology, the figures in Problem 7 are categorized in the same group. That means that it should be possible to change the sphere into a pancake or snake, etc. without cutting or tearing. Discuss how this could be accomplished. Stretching or scrunching and smoothing.

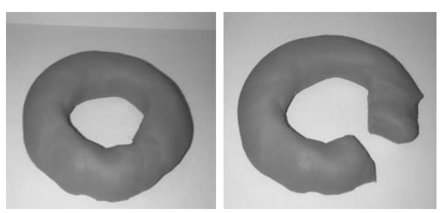
b. Use an un-bent paperclip or a plastic knife to gently etch a simple closed curve onto the surface of your figure; circles and ovals are recommended. Then slice the figure. (Think cross sections!)





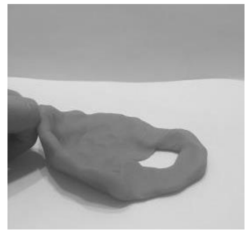
c. All of these figures were sliced into two pieces. In order to keep the figure in one piece, we can make zero slices, so these figures are classified as having a **genus** of 0. Sketch a picture of what the result of your slice looked like. See pictures above.

9. Dino wondered what would happen if he made his modeling clay into a donut.
   a. Does the donut behave in the same way as the pancake? Investigate with your team. Be sure to etch your closed curve before slicing. Note the observations your team makes. What is the genus of a donut? Justify your reasoning.

      The donut can be sliced parallel to the table to get two separate pieces. But the genus of a donut is 1 because one slice can be made without separating the figure into two pieces (see picture above).
   b. Can you think of other figures or items with the same genus? The letter P, a DVD, a coffee cup.

10. A topologist sees no difference between a coffee cup and a donut. Using modeling clay, work with your team to morph your donuts into coffee cups. Remember to change it without cutting or tearing! Check in with your teacher when finished.

MORE THAN DONUTS

11. What types of figures have a higher genus? Brainstorm with your team and sketch a few of your findings.
    a. Figures with a genus of 2. Scissors, the symbol for infinity, 8, glasses, a double inner tube, etc.

    b. Figures with a genus of 3, 4, 5, or higher. Pretzel, dining room table chair, doily, spaghetti strainer, chain link fence, etc.

CLOSURE

12. Let's revisit Problem 1. Anthony's alphabet grouping was by genus. Work with your team to place the remaining letters in the appropriate genus.
    a. Genus 1: {A, D, O, P, Q, R}
    b. Genus 2: {B}
    c. Genus 0: {C, E, F, G, H, I, J, K, L, M, N, S, T, U, V, W, X, Y, Z}

Lesson 4: Exploring Euler Numbers

Background Knowledge & Overview:

- Note: Any mention of a geometric "solid" in this lesson refers to the surface of the solid.
- ***Prior Knowledge:*** Students should be familiar with faces, vertices, and edges of solids. Students should also be familiar with Platonic solids (regular polyhedra) and finding surface area and volume of non-regular solids. For students who have not learned about regular polyhedra, see the note in the "Suggested Lesson Activity" section.
- ***Transition to New ideas:*** By analyzing the faces, vertices, and edges of three-dimensional solids, students will investigate the topological invariant of the Euler characteristic (Euler number) of sets of solids.
- ***Lesson Overview:*** Students will use Euler's formula to determine the Euler number for solids homeomorphic to the sphere and torus.

Goal:

- This lesson builds off of the previous lesson in the sense that surfaces of solids are being analyzed according to shape, not size. Through the analysis process, students are again blending algebra skills with geometry (and topological) content. This lesson uses counting techniques as a way to show that objects are topologically equivalent.

Objective:

- To develop Euler's formula and explore its implications for the surfaces of non-regular solids.

Standards:

- MP.3 – Construct viable arguments and critique the reasoning of others.
- MP.7 – Look for and make use of structure.
- MP.8 – Look for and express regularity in repeated reasoning.

Length of Activity:

- One class period (approximately 50 minutes)

Materials:

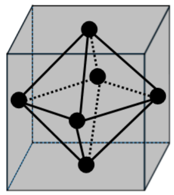
- Models
  - Option 1: Nets of Regular Polyhedra/Platonic Solids to build; tape required (add 15 mins to the lesson)
  - Option 2: solids for students to analyze (Platonic solids, pyramids, prisms, spheres, or modeling clay, etc.)
- Student Handout: Exploring Euler Numbers
- Teacher Answer Document: Exploring Euler Numbers

Suggested Lesson Activity:

Pairs or teams of four are appropriate for this lesson.

Note: If students are not familiar with Platonic solids, Question 1 could be replaced with each group cutting out one of the nets and building the solid. Groups would analyze their solid and then present the information to the class. At that time, the teacher could tell them the official name for the solid. To simplify the process even more, the figures could just be labeled *A*, *B*, *C*, *D*, and *E* in increasing order of the number of faces.

Start the class by having a student read the introduction and Question 1. Then give students a minute or two to brainstorm all that they remember about Platonic solids. Discuss the answers as a class and then fill in the first column of the table in Question 2. Then have teams work on Problem 2. For Problem 2a it is assumed that the concept of duals of Platonic solids have been covered in an earlier lesson. Duals are created by connecting the centers of all of the faces of a polyhedron. See the example below.



The cube and octahedron are duals of each other.

Students should check in with the teacher once they have completed part (d). Ensure that the Euler numbers are all 2 in the last column. Ask for students' predictions and justifications for the Euler number of non-regular polyhedra and then have them continue working.

Students should check in with the teacher after 2f and 4e. Read through Problem 5 together to ensure that students understand that each vertex of the hole on a base needs to be connected to the corresponding vertex on that same base.

Problems 5b and 6 can be done as a whole-class discussion. For Problem 5b, it may be difficult for the students to realize that there is exactly one face. To represent this fact, the donut can be created from one face using a piece of paper. Roll the paper into a cylinder. Hold the cylinder so that it is parallel to the ground and instruct students to imagine stretching and bending the cylinder so that the two circular bases connect.

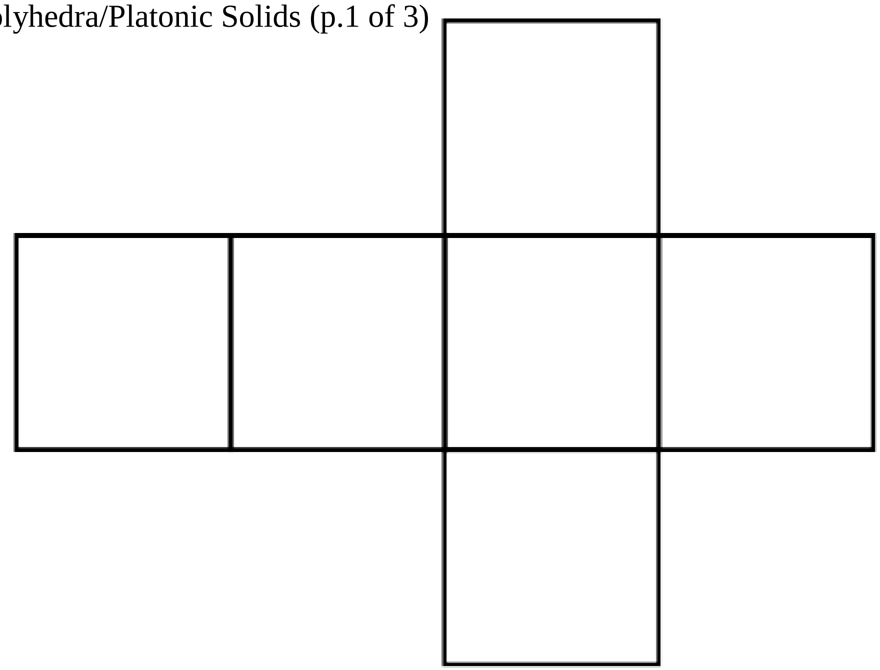
Closure:

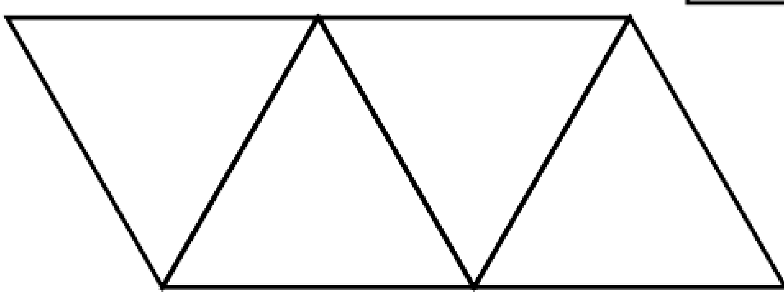
- Problem 6 is the closure problem for this lesson. The Euler number is a topological invariant which means that if two surfaces are homeomorphic (same shape) their Euler numbers will be equal. The Euler number is the same even though the solids are all geometrically different.

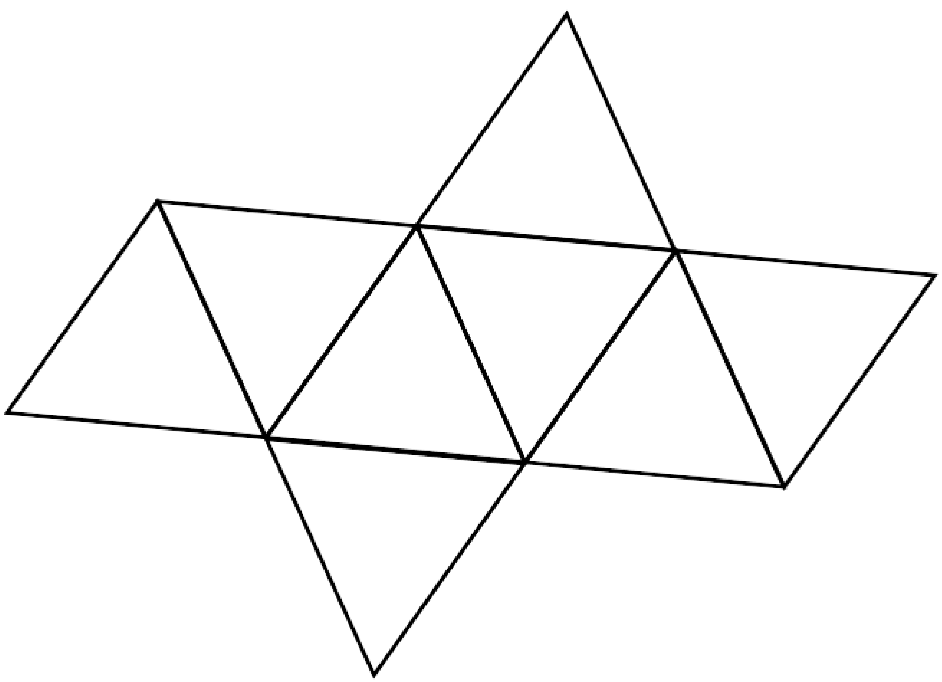
Homework Suggestion:

- Provide students with more examples of 3D objects that are homeomorphic to the sphere, torus, double-torus, etc. Have them verify the Euler number. For objects homeomorphic to the torus students can review calculating the surface area and volume.

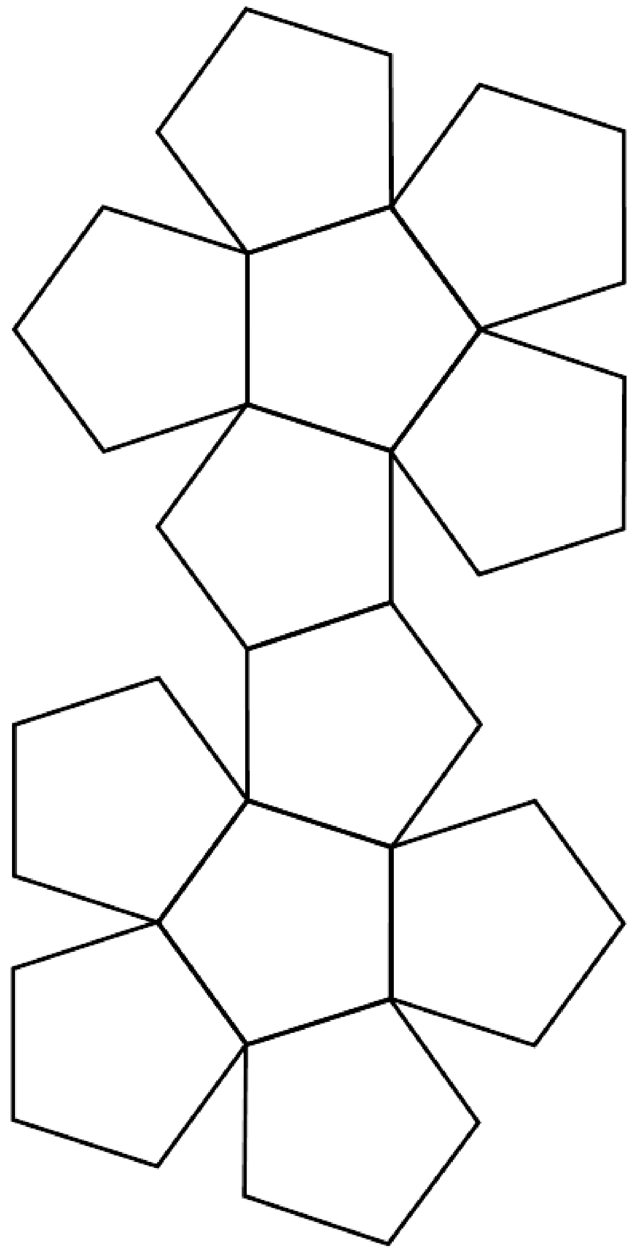
Nets of Regular Polyhedra/Platonic Solids (p.1 of 3)





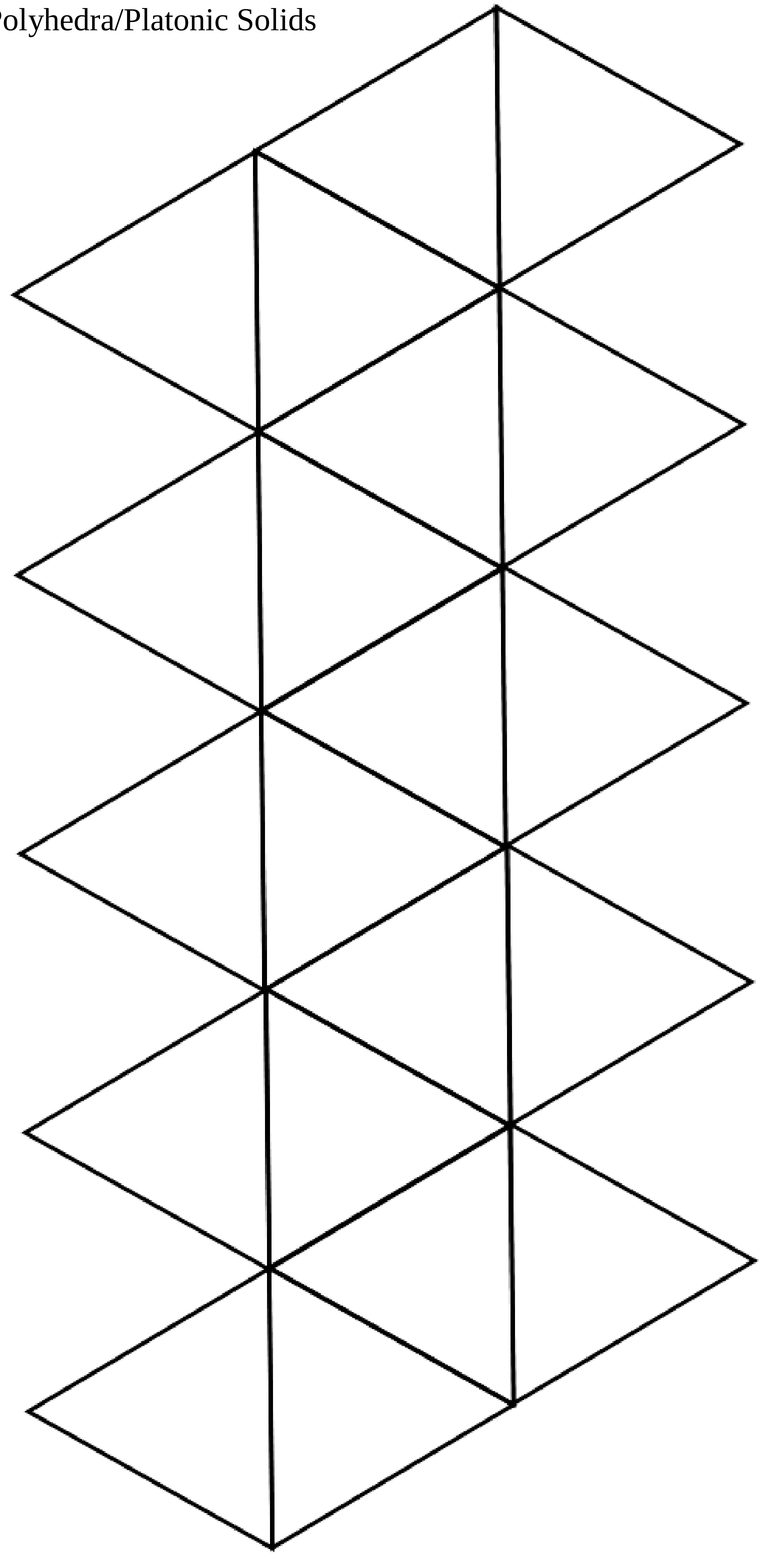


Nets of Regular Polyhedra/Platonic Solids (pg. 2 of 3)



Nets of Regular Polyhedra/Platonic Solids

(pg. 3 of 3)



Exploring Euler Numbers Name:______________________

Student Handout

In the previous lesson, we discussed how figures can be geometrically different, but topologically the same. Using Platonic solids as a starting point, we'll continue analyzing solids topologically while we investigate patterns of their properties.

Note: Any mention of the word "solid" in this lesson refers to the surface of the solid.

PLATONIC SOLIDS

1. Earlier in the course, we discussed the five Platonic solids, also known as regular polyhedra. List everything you remember about them and be ready to share with the class.

2. The vertices, edges, and faces of solids have a special relationship. To discover the relationship, fill out the table below. List the regular polyhedra in increasing order based on the number of faces.

Regular Polyhedron	**Vertices**	**Edges**	**Faces**	

   a. Look back at the data in the table. Do you see any patterns or notice any relationships?

   b. In the last column of the table, write $V - E + F$. Calculate the result for each regular polyhedra.

c. What repeated patterns or results did you find?

d. The formula $V - E + F$ is called Euler's (read "oiler's") formula and the result is referred to as an Euler number. Do you think the same result will occur when analyzing non-regular solids like pyramids and prisms? Justify your answer.

e. Test it out and record the results in the table.

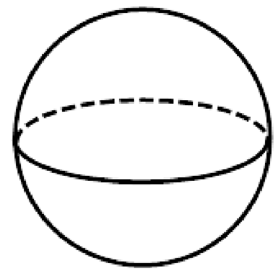
Solid	**Vertices**	**Edges**	**Faces**	**V-E+F**
Square Pyramid				
Pentagonal Pyramid				
Pentagonal Prism				
Hexagonal Prism				

f. Was your prediction correct?

BACK TO TOPOLOGY

3. Why does this result occur? The figures discussed so far are all considered to be equivalent to each other in topology because they are all homeomorphic to each other. Homeomorphic means they can all be changed into the same figure, just like how we turned a donut into a coffee cup. Using the solids from the data tables above, imagine if we poked a straw into each solid and inflated it. What would each figure become?

4. Olivia was confused. "How can we find the Euler number for a sphere? It doesn't have any flat surfaces." Hayden chimed in, "Maybe we could section off the surface of the sphere just like lines of latitude and longitude on a globe. By drawing the equator and two lines of longitude, it'll have 8 sections – those can be the faces."
    a. Draw Hayden's suggestion on the sphere at right.



    b. The vertices occur at the intersection of the lines. How many are there?

    c. As with other polyhedra, an edge is still a path that connects two vertices. How many are there?

    d. Verify that the Euler number for the sphere is 2.

    e. Olivia isn't fully convinced. "What if you section the sphere differently? Would the Euler number still be 2?" Experiment with your team.

MORE DONUTS

5. Gavin thought back to yesterday's lesson and asked, "What happens if we try to find the Euler number for a donut? Will it also be 2?"
   a. Let's first consider figures that are homeomorphic to a donut. Complete the table below.

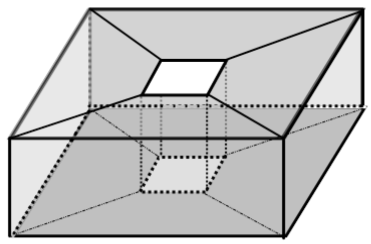


Figure 1

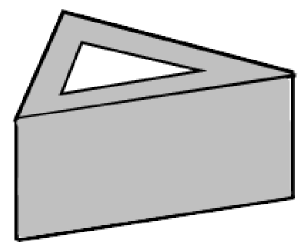
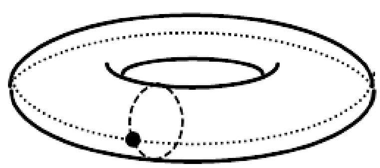


Figure 2

*Note: In each figure, each vertex of the hole and the corresponding vertex of the base (on the same plane) are connected by an edge.

Solid	**Vertices**	**Edges**	**Faces**	**V-E+F**
Figure 1				
Figure 2				

   b. From the chart in part (a), we know that the Euler number for a donut is _____. Using the diagram below, verify the Euler number for a donut.



CLOSURE

6. Compare the results of the Euler number for a sphere compared to a donut (called a **torus** in topology). What if we began working with solids like cement cinder blocks or double inner tubes (called a double torus)? Make a prediction of what the Euler number would be for those solids. Justify your conclusion.

Exploring Euler Numbers Name:________________________

Teacher Answer Document

In the previous lesson, we discussed how figures can be geometrically different, but topologically the same. Using Platonic solids as a starting point, we'll continue analyzing solids topologically while we investigate patterns of their properties.

Note: Any mention of the word "solid" in this lesson refers to the surface of the solid.

PLATONIC SOLIDS

1. Earlier in the course, we discussed the five Platonic solids, also known as regular polyhedra. List everything you remember about them and be ready to share with the class. During the class discussion, most of the following should come up: They are solids made up of congruent regular polygonal faces. Tetrahedron (4 triangular faces), Hexahedron/cube (6 square faces), octahedron (8 triangular faces), dodecahedron (12 pentagonal faces), icosahedron (20 triangular faces). Surface area is calculated by multiplying the area of one polygonal face times the number of faces.

2. The vertices, edges, and faces of solids have a special relationship. To discover the relationship, fill out the table below. List the regular polyhedra in increasing order based on the number of faces.

Regular Polyhedron	**Vertices**	**Edges**	**Faces**	**V-E+F**
Tetrahedron	4	6	4	2
Hexahedron/Cube	8	12	6	2
Octahedron	6	12	8	2
Dodecahedron	20	30	12	2
Icosahedron	12	30	20	2

   a. Look back at the data in the table. Do you see any patterns or notice any relationships? The hexahedron and octahedron have the same data, but the number of vertices and faces are switched. Same for the dodecahedron and icosahedron. This is because the figures are duals of each other.

b. In the last column of the table, write $V - E + F$. Calculate the result for each regular polyhedra. See table above.

c. What repeated patterns or results did you find? Every regular polyhedron had a result of 2.

d. The formula $V - E + F$ is called Euler's (read "oiler's") formula and the result is referred to as an Euler number. Do you think the same result will occur when analyzing non-regular solids like pyramids and prisms? Justify your answer. Answers vary; but yes, a tetrahedron is a pyramid and a cube is a prism. The Euler number should remain the same.

e. Test it out and record the results in the table.

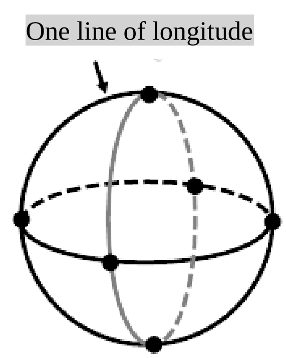
3D Figure	Vertices	Edges	Faces	V-E+F
Square Pyramid	5	8	5	2
Pentagonal Pyramid	6	10	6	2
Pentagonal Prism	10	15	7	2
Hexagonal Prism	12	18	8	2

f. Was your prediction correct? Answers vary.

BACK TO TOPOLOGY

3. Why does this result occur? The figures are all considered to be equivalent to each other in topology because they are all homeomorphic to each other. Homeomorphic means they can all be changed into the same figure. Just like how we turned a donut into a coffee cup. Using the solids from the data tables above, image if we poked a straw into each solid and inflated it. What would each figure become? A sphere.

4. Olivia was confused. "How can we find the Euler number for a sphere? It doesn't have any flat surfaces." Hayden chimed in, "Maybe we could section off the surface of the sphere just like lines of latitude and longitude on a globe. By drawing the equator and two lines of longitude, it'll have 8 sections – those can be the faces."



   a. Draw Hayden's suggestion on the sphere at right.

   b. The vertices occur at the intersection of the lines. How many are there? 6.

c. As with other polyhedra, an edge is still a path that connects two vertices together. How many are there? 12.

d. Verify that the Euler number for the sphere is 2. $V - E + F = 6 - 12 + 8 = 2$

e. Olivia isn't fully convinced. "What if you section the sphere differently? Would the Euler number still be 2?" Experiment with your team. Yes.

MORE DONUTS

5. Gavin thought back to yesterday's lesson and asked, "What happens if we try to find the Euler number for a donut? Will it also be 2?"

   a. Let's first consider figures that are homeomorphic to a donut. Complete the table below.

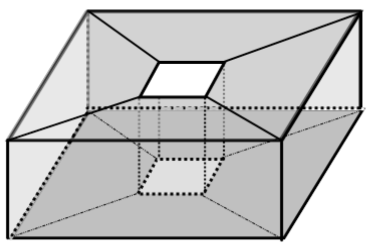


Figure 1

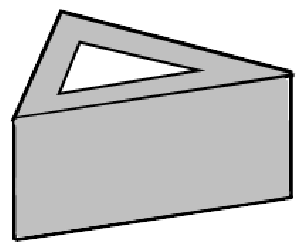
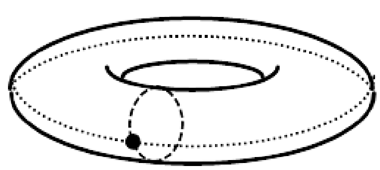


Figure 2

*Note: In each figure, each vertex of the hole and the corresponding vertex of the base (on the same plane) are connected by an edge.

3D Figure	Vertices	Edges	Faces	V-E+F
Figure 1	16	32	16	0
Figure 2	12	24	12	0

   b. From the chart above, we know that the Euler number for a donut is __0__. Using the diagram below, verify the Euler number for a donut.



$V - E + F = 1 - 2 + 1 = 0$

Vertices: 1

Edges: 2 (1 latitude, 1 longitude)

Faces: 1 (see the explanation in the "Suggested Lesson Activity")

CLOSURE

6. Compare the results of the Euler number for a sphere compared to a donut (called a **torus** in topology). What if we began working with solids like cement cinder blocks or double inner tubes (called a double torus)? Make a prediction of what the Euler number would be for those solids. Justify your conclusion. For each additional hole the Euler number decreases by 2. For instance, the Euler number or a double torus is -2 and the Euler number of a 3-holed torus is -4.

Lesson 5: Exploring Infinity

Background Knowledge & Overview:

- ***Prior Knowledge:***
  - Students should be familiar with the following regarding infinity: lines extend infinitely, there are an infinite number of points on a line; there are an infinite number of polygons; as the number of sides of regular polygon approaches infinity, the polygon looks like a circle.
  - Students should be comfortable with dimensional analysis, analyzing patterns in t-charts, and substitution and/or solving linear equations.
- ***Transition to New Ideas:*** Building off of students' general understanding of infinity, students can begin to explore mathematical philosophy by investigating infinity paradoxes.
- ***Lesson Overview:*** Students will use dimensional analysis, number sense, solving systems of equations and the visual aid of video clips to explore the complexities of infinity.

Goal:

- Students are familiar with the concept of infinity as it applies to points and lines in geometry. But there are complexities to infinity that – while mind-bending and abstract – are reasonable for high school students to be able to analyze and discuss. While the concept of infinity is not as heavily tied to geometry as the other lessons in this unit, I include it in my curriculum because it fosters abstract, critical thinking and has been an enjoyable and memorable lesson for my students. It also unofficially introduces infinite series and infinite sums which students will learn in depth in future mathematical courses.

Objective:

- To explore the concept of infinity and its paradoxes.

Standards:

- MP.1 – Make sense of problems and persevere in solving them.
- MP.2 – Reason abstractly and quantitatively.
- MP.3 – Construct viable arguments and critique the reasoning of others.

Length of Activity:

- One class period (approximately 50 minutes)

Materials:

- Optional: Dry Erase markers and boards (glossy desks work as boards)
- Calculators (Scientific capabilities, graphing not required)
- Student Handout: Exploring Infinity
- Teacher Answer Document: Exploring Infinity

Suggested Lesson Activity:

This lesson is intended for students to work in pairs.

Note: A teacher-led, handout-free discussion may work well for this lesson. Students might enjoy discussing and writing answers on dry erase boards as opposed to standard notes; this could increase student engagement since the content becomes abstract quickly.

Have a volunteer read the introduction to the lesson and Problem 1a. Give students time to talk in pairs and then call on a pair to state the answer. Then, the teacher should lead the discussion that occurs in the remainder of Problem 1.

Give students approximately three minutes to read and discuss Problem 2. As you hear students' thoughts and discussions, some probing questions are:

- If we take all of the odd numbers out of the counting numbers, are there more counting numbers than even numbers?
- If we include the negative numbers and zero, aren't there almost twice as many integers as counting numbers?
- What about fractions? Is there the same amount of fractions as counting numbers?

Problem 3 introduces Hilbert's Infinite Hotel. Note: At this point in my own classroom, I would show "Hilbert's Infinite Hotel – 60-second Adventures in Thought (4/6)" via YouTube as the initial explanation of the paradox [10]. The problem statement on the student handout is a summary of this video clip. Give students 8 minutes to think and discuss parts (a) and (b). Then, have a class discussion to hear students' thoughts. Part (c) is intended to be a teacher-led discussion to show students how "infinity + 1" and the even numbers each have a one-to-one correspondence to the set of counting numbers, and are therefore infinite sets of equivalent size.

Problem 5 introduces Zeno's dichotomy paradox. This question should be read as a class. After reading part (a), give students 1 minute to come up with their best guess for the definition of dichotomy. Then have a handful of student pairs state their guesses for the class. After hearing them all, the teacher should state the correct definition. Students can then complete part (b) in approximately one minute.

Problem 6 states the dichotomy as depicted in the book Philosophical Perspectives on Infinity [9]. Completing this problem as a class will allow for time to be spent on Problem 7, which has students prove that Achilles can travel the distance from *A* to *B* because the sum of the convergent infinite series is the distance he needs to travel (in this case, 120 feet). This problem is scaffolded so that pairs can try it independently, but it could also be done as a whole-class discussion. Alternatively, the discussion of Problem 7 can be covered with the YouTube clip entitled "Can You Solve Zeno's Paradox? – Brain Teaser" [7].

Closure:

- Below is the prompt for closure:
  - Today, we talked about the complexities of infinity. It can be pretty mind bending! Tell your partner one new thing you learned today and one thing you still have questions about.

Homework Suggestion:

- Research and summarize more of Zeno's infinity paradoxes (Achilles and the Tortoise, the Arrow, and the Stade).
- Provide your own analysis of any possible flaws in Zeno's thinking.

Exploring Infinity Name:_________________________

Student Handout

Lines extend indefinitely. Line segments are made of an infinite number of points. As the number of sides of a regular polygon approaches infinity, the polygon looks more and more like a circle. But what is infinity? Today, we will explore the concept of infinity.

BIGGER THAN…?

1. How big is infinity? First, think smaller. How big is 1 billion?
    a. What is 1 billion in number form? How many zeros does it have?
    b. How old is a person who's been alive for 1 billion seconds? You have 45 seconds to make an estimate with your partner. No phones or calculators may be used at this time! Be ready to share your guess with the class.
    c. Now take 3 minutes to calculate the age of a person who's been alive for 1 billion years. Be sure to show your work. Has anyone in this room been alive for 1 billion seconds?
    d. The average person world-wide lives to 70 years of age, approximately how many seconds do they live? Approximately how many seconds have you been alive?
    e. 1 billion seems like a large quantity – and it is! – but infinity is even bigger! Consider all the grains of sand on beaches and in the ocean PLUS all the hairs on every person's head PLUS all the stars of the galaxy. Infinity is BIGGER! How would you define infinity?

TO INFINITY…AND BEYOND?

2. Are all infinities the same size? Consider the following infinities as you answer the question. Be ready to justify your reasoning to the class.
    - All the counting numbers
    - All the even numbers
    - All the integers
    - All the decimals between 0 and 1.

HILBERT’S INFINITE HOTEL

3. Studying infinity in this way is “new” in mathematics. Georg Cantor (1845-1918 CE) was the first mathematician to study infinity theories, specifically in the realm of set theory. Not long after Cantor’s death, David Hilbert presented the Infinite Hotel Paradox. It may seem like these mathematicians lived long ago, but since the beginnings of mathematics occurred in ancient Mesopotamia approximately 4000 years ago, relatively speaking, 100 years doesn’t seem so long ago!

    Hilbert’s Infinite Hotel is described below. Work through parts (a) and (b) with your team. The entire class will do part (c) together with your teacher.

    a. Imagine a hotel that has an infinite number of rooms and no vacancies – it’s 100% full. What happens when a new guest arrives? Hilbert’s solution is: Each guest can move into the room number that is one larger than their current room. Then, room 1 will be available for the new guest. So does $\infty + 1 = \infty$? Discuss this situation with your team and write your thoughts below. Is there a way to show that infinity and “infinity + 1” are the same size?

    b. What happens if an infinitely large bus comes with infinitely more guests? The hotel is full – is it possible to give each guest a room? Hilbert’s solution is: Yes! All guests can be accommodated by making each current guest move to the number that is twice as big as theirs. This leaves an infinite amount of odd-numbered rooms available for guests on the infinite bus. So is $\infty + \infty = \infty$? Discuss this situation with your team. Is there a way to show that two sets of infinity together are the same size as one set of infinity?

c. *To be completed with your teacher.* Are all of these infinities the same size??

ADVERTISING FOR HILBERT'S HOTEL

4. If a marketing advertiser came up with the tagline:

   "Hilbert's Infinite Hotel: Never any vacancy, but always room for more."

   Is this honest marketing?

ZENO'S DICHOTOMY PARADOX

5. Other famous infinity paradoxes are puzzles written by the philosopher Zeno of Elea (490 BCE). The word dichotomy may sound familiar if you have used a dichotomous key to classify leaves or other types of plants or animals in a science course.
    a. What do you think the word dichotomy means?

    b. Zeno's dichotomy paradox discusses traveling the distance between two points *A* and *B*. Sketch the line segment below that depicts the distance to be traveled. If we're considering the dichotomy of this geometric object, what is the geometry vocabulary term that has the same meaning as dichotomy?

6. Zeno's dichotomy paradox: Suppose that Achilles is standing at point *A* and wants to travel to point *B* which is 120 feet away. To do this, he must first run to a point that is 60 feet away. But to get there, he must first run to a point 30 feet away. But to do that, he must run to a point that is 15 feet away. And so on. Is it possible for Achilles to even start moving?

7. We know that motion <u>is</u> possible, so let's consider a variation of the dichotomy paradox. Achilles is still trying to get from point *A* to point *B*, which spans 120 feet. Assume that Achilles starts at point *A* and successfully travels 60 feet to the midpoint of *A* and *B*. Label the point *W*. Then, Achilles travels half the distance between *W* and B. Label the point *X*. Continue the pattern.
    a. Sketch a picture in the space below to illustrate the above situation.

b. Does it appear that Achilles ever reaches point *B*?

c. Let's investigate the situation algebraically. The distance Achilles travels is a sum of an infinite number of bisected distances. Follow the prompts below to show that it is possible for Achilles to travel the 120 feet to get from point *A* to point *B*.

   i. Let *S* stand for sum.

   $S =$ _____ + _____ + _____ + _____ + _____ + …

   ii. Multiply both sides of the equation by 2.

   $2S =$ _____ + _____ + _____ + _____ + _____ + …

   iii. The equation in part (ii) has the same infinite tail as *S*. Use substitution to rewrite the equation for 2*S*.

   $2S =$ _____ + _____

   iv. Solve for *S* in the space above. You just showed that Achilles can, in fact, travel the distance from *A* to *B*!

Exploring Infinity Name:________________________

Teacher Answer Document

Lines extend indefinitely. Line segments are made of an infinite number of points. As the number of sides of a regular polygon approaches infinity, the polygon looks more and more like a circle. But what is infinity? Today, we will explore the concept of infinity.

BIGGER THAN…?

1. How big is infinity? First, think smaller. How big is 1 billion?
   a. What is 1 billion in number form? How many zeros does it have? 1,000,000,000; 9 zeros.

   b. How old is a person who's been alive for 1 billion seconds? You have 45 seconds to make an estimate with your partner. No phones or calculators may be used at this time! Be ready to share your guess with the class. Most of my students guessed between 1-10 years.

   c. Now take 3 minutes to calculate the age of a person who's been alive for 1 billion years. Be sure to show your work. Has anyone in this room been alive for 1 billion seconds?

$$\text{time} = \frac{1000000000\,\text{sec}}{1}\left(\frac{1\text{ minute}}{60\text{ seconds}}\right)\left(\frac{1\text{ hour}}{60\text{ minutes}}\right)\left(\frac{1\text{ day}}{24\text{ hour}}\right)\left(\frac{1\text{ year}}{365\text{ days}}\right)$$
$$\approx 31.7\text{ years}$$

   No student in our high school has been alive for 1 billion seconds.

   d. The average person world-wide lives to 70 years of age, approximately how many seconds do they live? Approximately how many seconds have you been alive? Between 2 billion and 2.5 billion seconds; a 15 year old is approximately 0.5 billion seconds old.

   e. 1 billion seems like a large quantity – and it is! – but infinity is even bigger! Consider all the grains of sand on beaches and in the ocean PLUS all the hairs on every person's head PLUS all the stars of the galaxy. Infinity is BIGGER! How would you define infinity?
   Wolfram Mathworld defines it as "an unbounded quantity that is greater than every real number" [13]. Merriam Webster's Dictionary defines it as "an indefinitely great number or amount" [3].

TO INFINITY…AND BEYOND?

2. Are all infinities the same size? Consider the following infinities as you answer the question. Be ready to justify your reasoning to the class.
    - All the counting numbers (0, 1, 2, 3, 4, …)
    - All the even numbers (0, 2, 4, 6, 8, …)
    - All the integers (…-2, -1, 0, 1, 2, …)
    - All the decimals between 0 and 1. They can't be put into a list; and therefore are a larger infinity than the countable infinities above.

HILBERT'S INFINITE HOTEL

3. Studying infinity in this way is "new" in mathematics. Georg Cantor (1845-1918 CE) was the first mathematician to study infinity theories, specifically in the realm of set theory. Not long after Cantor's death, David Hilbert presented the Infinite Hotel Paradox. It may seem like these mathematicians lived long ago, but since the beginnings of mathematics occurred in ancient Mesopotamia approximately 4000 years ago, relatively speaking, 100 years doesn't seem so long ago!

    Hilbert's Infinite Hotel is described below. Work through parts (a) and (b) with your team. The entire class will do part (c) together with your teacher.

    a. Imagine a hotel that has an infinite number of rooms and no vacancies – it's 100% full. What happens when a new guest arrives? Hilbert's solution is: Each guest can move into the room number that is one larger than their current room. Then, room 1 will be available for the new guest. So does $\infty + 1 = \infty$? Discuss this situation with your team and write your thoughts below. Is there a way to show that infinity and "infinity + 1" are the same size? Yes. See the answer in part (c).

    b. What happens if an infinitely large bus comes with infinitely more guests? The hotel is full – is it possible to give each guest a room? Hilbert's solution is: Yes! All guests can be accommodated by making each current guest move to the number that is twice as big as theirs. This leaves an infinite amount of odd-numbered rooms available for guests on the infinite bus. So is $\infty + \infty = \infty$? Discuss this situation with your team. Is there a way to show that two sets of infinity together are the same size as one set of infinity? Yes. See the answer in part (c).

c. Are all of these infinities the same size?? Yes. Infinities are the same size if there is a 1-to-1 correspondence between them. Think back to algebra, each linear function has a 1-to-1 correspondence because each input (x-value) has exactly one output (y-value).

Think of it this way: Without counting, how can we know if both of your hands have the same number of fingers? Pair them up! We don't need to know the number of fingers on each hand; we just need to know that each hand has the same amount.

We can show that the original guests in the hotel are the same size infinity as when a new guest came. The one-to-one correspondence is as follows:

Full hotel	*n*	1	2	3	4	5	…
Full hotel + 1 new guest	*n+1*	2	3	4	5	6	…

Similarly, we can show that the original guests in the hotel – the counting numbers – are the same size infinity as their new room numbers – the even numbers – because of the following one-to-one correspondence:

Counting numbers	*n*	1	2	3	4	5	…
Even numbers	*2n*	2	4	6	8	10	…

Further, since the full hotel is the same size infinity as both the even numbers and the "full hotel + 1," all three sets of infinity are the same size.

ADVERTISING FOR HILBERT'S HOTEL

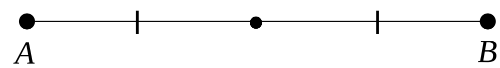
4. If a marketing advertiser came up with the tagline:

"Hilbert's Infinite Hotel: Never any vacancy, but always room for more."

Is this honest marketing? This tagline is good marketing. Since people are countable entities, no matter how many new people arrive, there will always be a one-to-one correspondence of the "full hotel plus the new guests" to the counting numbers.

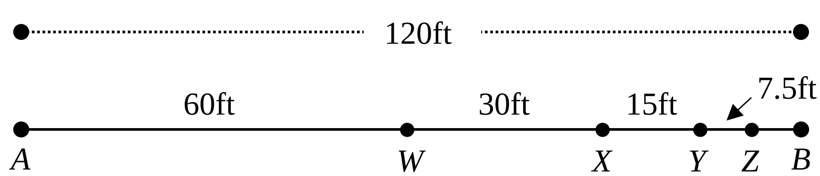
ZENO'S DICHOTOMY PARADOX

5. Other famous infinity paradoxes are puzzles written by the philosopher Zeno of Elea (490 BCE). The word dichotomy may sound familiar if you have used a dichotomous key to classify leaves or other types of plants or animals in a science course.
    a. What do you think the word dichotomy means? Dichotomy means to split into two distinct groups.
    b. Zeno's dichotomy paradox discusses traveling the distance between two points *A* and *B*. Sketch the line segment below that depicts the distance to be traveled. If we're considering the dichotomy of this geometric object, what is the geometry vocabulary term that has the same meaning as dichotomy? Bisect.



6. Zeno's dichotomy paradox: Suppose that Achilles is standing at point *A* and wants to travel to point *B* which is 120 feet away. To do this, he must first run to a point that is 60 feet away. But to get there, he must first run to a point 30 feet away. But to do that, he must run to a point that is 15 feet away. And so on. Is it possible for Achilles to even start moving? Based on Zeno's assumption that space and time are points on the real number line, no. There is no "smallest" midway point for him to travel to because the distance can be bisected an infinite number of times.

7. We know that motion <u>is</u> possible, so let's consider a variation of the dichotomy paradox. Achilles is still trying to get from point *A* to point *B*, which spans 120 feet. Assume that Achilles starts at point *A* and successfully travels 60 feet to the midpoint of *A* and *B*. Label the point *W*. Then, Achilles travels half the distance between *W* and B. Label the point *X*. Continue the pattern.
    a. Sketch a picture in the space below to illustrate the above situation.



    b. Does it appear that Achilles ever reaches point *B*? No. He will be very close, the remaining distance left to travel can be bisected an infinite number of times.

c. Let's investigate the situation algebraically. The distance Achilles travels is a sum of an infinite number of bisected distances. Follow the prompts below to show that it is possible for Achilles to travel the 120 feet to get from point *A* to point *B*.

   i. Let *S* stand for sum.
   $S = 60 + 30 + 15 + 7.5 + 3.75 + \ldots$

   ii. Multiply both sides of the equation by 2.
   $2S = 120 + 60 + 30 + 15 + 7.5 + \ldots$

   iii. The equation in part (ii) has the same infinite tail as *S*. Use substitution to rewrite the equation for 2*S*.

   $$2S = 120 + S$$
   $$S = 120$$

   iv. Solve for *S* in the space above. You just showed that Achilles can, in fact, travel the distance from *A* to *B*! See above in part (iii).

Conclusion

My goal in creating these lessons is to ensure that I am providing content-rich opportunities in my Honors Geometry classes beyond the depth of learning that is tested by the state. In my experience, students at the honors level are more intrinsically motivated to think critically and abstractly about mathematical concepts. As such, I've added lessons that I think will whet their appetites, stretch their abstract thinking skills, and ensure that they have a more accurate view of mathematics as a whole. I've been able to test-run some of these lessons in previous classes, and some of those students still reminisce excitedly about the 'bonus' topics that we discussed.

In alignment with the description of rigor from the van Hiele Levels of Geometric Understanding, these lessons provide students with opportunities to establish and compare mathematical systems. In Lesson 1, the concepts of geometric transformations are explored through the algebraic structure of a group. Lesson 2 establishes the structure students have been operating with throughout their Euclidean geometry course and contrasts it to the structure of spherical geometry. Lessons 3 and 4 shift students' thinking from "same shape, different size" to "shape, not size" as attention to measurement is eliminated. In doing so, students explore the foundation of topology and investigate a topological invariant while discovering Euler's formula. The fifth lesson establishes the difference between large finite quantities and infinite quantities and delves into the complexities of infinity. It is my hope that with these lessons students will engage in conversations about upper level mathematics, become excited about the complex possibilities, and use their excitement to fuel their own investigations. In doing so, I believe the appropriate level of rigor for these students can be attained.

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