DISCOVERING CALCULUS WITH THE TI-NSPIRE CAS CALCULATOR

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DISCOVERING CALCULUS
WITH THE TI-NSPRIE CAS CALCULATOR

An Essay Submitted to the
Office of Graduate Studies
College of Arts & Sciences of
John Carroll University
In Partial Fulfillment of the Requirements
For the Degree of
Master of Arts

By
Meghan A. Nielsen
2017
The essay of Meghan A. Nielsen is hereby accepted:

__________________________________________
Advisor – Barbara K. D’Ambrosia

__________________________________________
I certify that this is the original document

__________________________________________
Author – Meghan A. Nielsen
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Introduction

I have been teaching calculus for the last five years. I am always looking for ways to make my lessons more interactive for students because I know student engagement is key to understanding and retaining mathematical material. In order to get students more involved in their learning, student-centered discovery lessons are essential. Also, using technology is a great way to keep students engaged in their learning. Therefore, I chose to design discovery lessons that use the TI-Nspire CAS calculators so students can explore calculus topics and formulate their own ideas while discovering formulas and/or theorems.

I used these lessons in my Advanced Placement Calculus AB class throughout the school year 2016-2017. This class consisted of twenty-nine students with a mix of juniors and seniors. Students in this course are advanced in their mathematical abilities over the majority of their peers. Therefore, these students are persistent in working through math problems and are more willing to try different approaches. Due to the intellectual curiosity and drive of these students, these lessons use some scaffolding in order to keep the students focused on the task at hand, but also give the students room to try their own ideas to discover how these calculus topics work.

The lessons that I created span the entire Advanced Placement Calculus AB course, which covers one school year at the high school level and is equivalent to a first semester college calculus course. These lessons work best if students are working in groups. I had my students in groups of three or four. The lessons consist of discovering the Power Rule, the Product and Quotient Rules, Velocity and Other Rates of Change, the Chain Rule, the Mean Value Theorem, and Slope Fields. In a traditional classroom, these ideas as just presented to the students. The students are then expected to memorize these ideas and apply them to problems. While this was my approach for the first four years of teaching calculus, I knew that I could make these calculus concepts more meaningful to students. I wanted to pique students’ interests and curiosity to see what they could discover while using a guided lesson along with the TI-Nspire CAS calculators. I was very happy with the results and believe that teachers should be using these discovery lessons more in their classrooms.

Below I have included my review of each lesson, along with the student document, the teacher document, and screenshots of the TI-Nspire files and Quick Polls for each lesson. The Quick Poll on the TI-Nspire CAS calculators allows teachers to obtain a quick formative assessment to help determine if students understand the mathematical concepts at the end of the class or at the beginning of the following class. This helps to ensure that students really understand what they have discovered and allows the teacher collect data.
Lesson 1: Discovering the Power Rule

Students were expected to discover the Power Rule for derivatives and then work through some examples with their group. As students were working through the lesson, I circulated around the room and observed how they were doing with the material.

Originally, I only had students work through one example where they calculated the derivative of a function using the definition of a derivative before asking them if they knew what the shortcut was. When I have taught this concept in previous years, it was in a more traditional format. I only used one example because we were having a whole class discussion. I led them to the rule instead of them discovering it on their own. Therefore, once I saw how the students were struggling with the lesson, I quickly realized that students needed more than one example in order to determine the shortcut. I told my students to try finding the derivative of more polynomials using the TI-Nspire CAS calculator. Then using those examples, I had them see if they could find the pattern. By having my students do more examples, every group was able to determine the pattern.

Due to that issue, I have since edited the lesson so that students will work out the example by hand and then fill in a table using the TI-Nspire CAS Calculator to ensure that they can find the pattern. In the table, I include functions with positive, negative, and fractional exponents so that students can see how the pattern is the same for any real power. I believe that this will allow students to be more successful while still discovering the Power Rule without being told directly what it is.

Also, my students did not have time to finish the entire lesson with the Quick Poll in one class period. Because of this, I took part of the next day to finish this lesson. I have since edited the lesson by eliminating a redundant practice problem, so hopefully students can finish in one class period. If there is still not enough time to finish the lesson in one class, I highly recommend having students complete the entire worksheet even if it takes part of another day.

I chose the practice problems very meticulously. I do not believe that any of the other practice problems can be eliminated. I started with students finding the derivative of a polynomial, which should be quick and easy. The second problem is more difficult because students must first rewrite the equation to have negative and fractional exponents, which is a skill students should have from a previous mathematics course, before applying the Power Rule. The third problem takes the process a step further because students must find the equations for the tangent and normal lines. This is a skill that my students learned in the previous unit pertaining to limits, so I used the problem to check if students could combine this new material with previous material. When choosing the functions and the points for the tangent and normal lines, I chose points where the derivative is neither 1 or \(-1\). This ensures that I can tell if students used a valid method to find the slope of the normal line.

I did not only include computational questions; I also included conceptual questions. The fifth problem is an example of this. Students need conceptual questions to make sure they are thinking about what a derivative really is. Conceptual problems are
more difficult because students have to write a response instead of just computing answers. This is a skill that is of upmost importance. The last question is different than anything my students had seen. While it may not be as challenging as the previous problem, it makes students think about how a tangent line is beneficial for approximating functions. This concept is used throughout the course and in higher mathematics, so I wanted to introduce students to this idea early on in the course.
Power Rule – TEACHER DOCUMENT (Activity Overview)

Required Time:
- 1 class (60 minutes)

Standards
- II. Derivatives
  - Concept of the derivative
  - Derivative at a point

Objective
- I can calculate derivatives in order to determine the derivative at a point and the equation of the tangent and normal line.

Materials
- Student Resources:
  - Power Rule (Student Document): 1 per student
  - TI-Nspire CAS Calculators: 1 per student (or at least 1 per group)
- Teacher Resources:
  - Power Rule – Teacher Document
  - TI-Nspire Navigator Software
  - Power Rule Quick Poll file

Activity Overview (about 50 minutes)
- Students work in groups of three to four to complete the Power Rule (Student Document).
- Circulate around the room to ensure students are on task and are progressing through the worksheet.
- Refrain from telling them the pattern for the Power Rule. If students are struggling to see the pattern from the table, direct students to try to see a pattern with the powers. It might be helpful to have students rewrite the answers that the TI-Nspire calculates in the table to not include any radicals or x’s in the denominator. In other words, encourage students to have negative and fractional exponents to see the pattern.
- Make sure that students are showing their work on problems 1-7 and then checking on the TI-Nspire.
- It is suggested that students are not allowed to use CAS calculators on their upcoming assessments. CAS calculators are permitted on the AP Calculus Exam, but students need to gain an understanding of the basics since part of the AP Calculus Exam is non-calculator. Also, students may take subsequent courses in which they are not allowed to use CAS calculators.

Closure (about 10 minutes)
- Have students answer the question, “What important points did you learn today?”
• Make sure students talk about what a derivative is, different ways to represent the derivative, the Power Rule, etc.

• During the last 5 minutes, send students the Quick Poll (Power Rule).
  o Have students complete this on their own.
  o If time allows, go over the question by showing the results and having a student explain the answer.
  o Note that the Quick Poll will accept any answers that are equivalent to the answer, so students do not need to simplify their answers.

Homework: Assign practice problems where students must apply the Power Rule to find the derivative of a function, derivative at a point, equation of the tangent line, and equation of the normal line.
Power Rule (Student Document)

Name: ______________________

Recall from the last chapter:

\[ m_{\text{tan}} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \]

Where that limit exists, it is also known as the derivative of a function!

There are four different ways to express the derivative: \( y', f'(x), \frac{dy}{dx}, \frac{d}{dx}(y) \).

The word differentiable means that the function has a derivative:

\[ f'(x) = m_{\text{tan}} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \]

Can we find a shortcut? Consider \( f(x) = x^3 \). Find \( f'(x) \) using the limit definition (not the TI-Nspire) and compare it to \( f(x) \).

Now check your answer on your TI-Nspire. To do so from the home screen, go to “Calculate” → “Menu” → “4: Calculus” → “Derivative”. Type in \( \frac{d}{dx}( ) \) and the function. Use your TI-Nspire to fill in the table listed below:

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( f'(x) )</th>
<th>( f(x) )</th>
<th>( f'(x) )</th>
<th>( f(x) )</th>
<th>( f'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 )</td>
<td>( x^{-4} )</td>
<td>( x^{1/2} )</td>
<td>( x^{1/2} )</td>
<td>( x^{1/2} )</td>
<td>( x^{1/2} )</td>
</tr>
<tr>
<td>( x^3 )</td>
<td>( x^{-5} )</td>
<td>( x^{-1/3} )</td>
<td>( x^{-1/3} )</td>
<td>( x^{-1/3} )</td>
<td>( x^{-1/3} )</td>
</tr>
<tr>
<td>( 5x^2 )</td>
<td>( 2x^{-4} )</td>
<td>( 6x^{1/2} )</td>
<td>( 6x^{1/2} )</td>
<td>( 6x^{1/2} )</td>
<td>( 6x^{1/2} )</td>
</tr>
<tr>
<td>( 5x^3 )</td>
<td>( 2x^{-5} )</td>
<td>( 6x^{-1/3} )</td>
<td>( 6x^{-1/3} )</td>
<td>( 6x^{-1/3} )</td>
<td>( 6x^{-1/3} )</td>
</tr>
</tbody>
</table>
Complete the formula from your observations above:

**Power Rule**: Let \( n \) be a real number. Then

\[
\frac{d}{dx} (x^n) = \frac{n}{x^{n-1}}
\]

1) Test your conjecture for the Power Rule by creating a polynomial and testing your rule! Make sure to write your original function and work out the derivative. Then check your answer by using the TI-Nspire CAS calculator.

For questions 2-4, use the Power Rule along with rules you already know (e.g. sum and difference rules) to find the derivative of each function. Compute the derivatives by hand (show your work), and then check on the TI-nspire. WARNING: You might have to do some algebra on the Nspire’s results to get it to match your answer!

2) \( y = 4x^3 + 5x^2 - 7 \)

3) \( y = \frac{1}{x^2} + 3x - \sqrt[3]{x} + 100 \)

4) Find the derivative of the function \( f(x) = 5\sqrt{x} + \frac{1}{\sqrt{x}} = 3\sqrt{x} + 2 \) at \( x = 64 \).

5) Find the lines that are tangent and normal to the curve \( y = \frac{1}{4}x^2 - 2x \) at \( x = 10 \).

6) Let \( f(x) = C \) where \( C \) is a real number. Why is \( f'(x) = 0 \)?
7. a) Find the equation of the tangent line to the curve $f(x) = x^3 + 2x$ at $x = 2$.

b) Use your equation of the tangent line to approximate the value of $f(1.999)$ without using a calculator!

c) Now use your calculator to find the value of $f(1.999)$.
Recall from the last chapter:

\[ m_{\tan} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}. \]

Where that limit exists, it is also known as the derivative of a function!

There are four different ways to express the derivative: \( y' \), \( f'(x) \), \( \frac{dy}{dx} \), \( \frac{d}{dx}(y) \).

The word **differentiable** means that the function has a derivative:

\[ f'(x) = m_{\tan} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}. \]

Can we find a shortcut? Consider \( f(x) = x^3 \). Find \( f'(x) \) using the limit definition (not the TI-Nspire) and compare it to \( f(x) \).

\[
\begin{align*}
f'(x) &= \lim_{h \to 0} \frac{(x + h)^3 - x^3}{h} \\
&= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\
&= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h} \\
&= \lim_{h \to 0} (3x^2 + 3xh + h^2) \\
&= 3x^2
\end{align*}
\]

Now check your answer on your TI-Nspire. To do so from the home screen, go to “Calculate” → “Menu” → “4: Calculus” → “Derivative”. Type in \( \frac{d}{dx}(f(x)) \) and the function. Use your TI-Nspire to fill in the table listed below:
Complete the formula from your observations above:

**Power Rule:** Let $n$ be a real number. Then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

1) Test your conjecture for the Power Rule by creating a polynomial and testing your rule! Make sure to write your original function and work out the derivative. Then check your answer by using the TI-Nspire CAS calculator.

**Answers will vary.**

For questions 2–4, use the Power Rule along with rules you already know (e.g. sum and difference rules) to find the derivative of each function. Compute the derivatives by hand (show your work), and then check on the TI-nspire. WARNING: You might have to do some algebra on the Nspire’s results to get it to match your answer!

2) $y = 4x^3 + 5x^2 - 7$

$$y' = 12x^2 + 10x$$

3) $y = \frac{1}{x^2} + 3x - \sqrt[3]{x} + 100$

$$y' = -2x^{-3} + 3 - \frac{1}{3}x^{-2/3} = -2x^{-3} + 3 - \frac{1}{3\sqrt[3]{x^2}}$$

4) Find the derivative of the function $f(x) = 5\sqrt[3]{x} + \frac{1}{\sqrt{x}} - \sqrt[3]{x} + 2$ at $x = 64$.

$$f'(x) = \frac{5}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} - \frac{1}{3}x^{-2/3}$$

and $f'(64) = \frac{893}{3072}$
5) Find the lines that are tangent and normal to the curve \( y = \frac{1}{4} x^2 - 2x \) at \( x = 10 \).

\[
y' = \frac{1}{2} x - 2
\]

Tangent Line: \( y = 3(x - 10) + 5 \)

\[
y'(10) = 3 = m_{\text{tan}}
\]

Normal Line: \( y = -\frac{1}{3}(x - 10) + 5 \)

6) Let \( f(x) = C \) where \( C \) is a real number. Why is \( f'(x) = 0 \)?

The derivative is a formula for the slope of the tangent line. A function that is a constant number is a horizontal line, which has a slope of zero everywhere. Thus, \( f'(x) = 0 \).

7. a) Find the equation of the tangent line to the curve \( f(x) = x^3 + 2x \) at \( x = 2 \).

\[
f'(x) = 3x^2 + 2
\]

\[
f'(2) = 14
\]

\[
f(2) = 12
\]

Tangent Line: \( y = 14(x - 2) + 12 \)

b) Use your equation of the tangent line to approximate the value of \( f(1.999) \) without using a calculator!

\[
y(1.999) = 14(1.999 - 2) + 12
\]

\[
= 14(-0.001) + 12
\]

\[
= -0.014 + 12
\]

\[
= 11.986
\]

c) Now use your calculator to find the value of \( f(1.999) \).

\[
f(1.999) = 11.986006
\]
Power Rule Quick Poll Screenshots:

Find the derivative of the following function:

\[ f(x) = 5x^2 + 12x \]

10 \cdot x + 12

Find the equation of the tangent line to the following function at the given point:

\[ f(x) = 10\sqrt{x} \text{ at } x = 16 \]

\[ y = \frac{5}{4} (x - 16) + 40 \]
Lesson 2: Discovering the Product and Quotient Rules

Students were expected to discover the Product and Quotient Rules for derivatives and then work through some examples with their group. All groups were able to find the pattern for the Product Rule with only minor difficulty. The Quotient Rule, though, was much more difficult. I gave everyone the hint as shown on the worksheet, but still only half of the groups found the pattern. Some students were very determined and were thrilled when they found the pattern successfully. Many of the students, though, were getting frustrated and just wanted me to tell them.

Originally, I gave this lesson to my students before they knew the derivatives of the trigonometric functions. Therefore, I formatted the table for the Product and Quotient Rules to use one function evaluated at different points. Students then had a numerical answer for the derivative at a point, and they had to combine the values of \( f(x) \), \( g(x) \), \( f'(x) \), and \( g'(x) \) at the different points to obtain \( y'(x) \) at that point. This proved to be more challenging than I expected, though, because the TI-Nspire CAS calculator simplified the final answer for \( y'(x) \) at the specific points. Most students were not able to identify the formula for the derivative of the quotient.

Since this lesson was a struggle for many students, I have reordered the unit so that students will first learn the derivatives of the trigonometric functions and then learn the Product and Quotient Rules. This allowed me to change the table to involve different functions for each row instead of using different values for \( x \). Therefore, the derivative that the students obtain from the TI-Nspire CAS calculators will show the \( f(x) \) and \( g(x) \) functions in the answer. This change should make it easier for students to discover the pattern for the Product and Quotient Rule than it was in the original lesson. This hopefully will prevent students from getting too frustrated with the lesson and help them to persevere and find the pattern.

The students were able to successfully apply both the Product and Quotient Rules in the practice problems that followed. I think after exploring this concept with the TI-Nspire CAS Calculators, students will more likely remember when the Product and Quotient Rules must be applied.

Also, my students did not have time to finish the entire lesson nor the Quick Poll in one class period. Since editing the lesson to use the trigonometric functions, I think students will discover the pattern more quickly. I would recommend having the students complete the Quick Poll the following day as an entrance ticket, though, because I do not believe there will be enough time to complete it as an exit ticket.
Product and Quotient Rule – TEACHER DOCUMENT (Activity Overview)

Required Time
- 1 class (60 minutes)

Standards
- II. Derivatives
  - Concept of the derivative
  - Derivative at a point

Objective
- I can calculate derivatives using the product rule and the quotient rule in order to determine the derivative at a point and apply derivatives to application problems.

Materials
- Student Resources:
  - Product and Quotient Rules (Student Document): 1 per student
  - TI-Nspire CAS Calculators: 1 per student (or at least 1 per group)

- Teacher Resources:
  - Product and Quotient Rules – Teacher Document
  - TI-Nspire Navigator Software
  - Product and Quotient Rule Quick Poll file

Activity Overview (about 55 minutes)
- Students work in groups of three to four to complete the Product and Quotient Rules (Student Documents). Hand out only the Product Rule worksheet at the beginning.
- Once groups are finished and have been checked on the Product Rule worksheet, give them the Quotient Rule worksheet.
- Circulate around the room to ensure students are on task and are progressing on the worksheet.
- Refrain from telling them the pattern for the Product Rule.
- Make sure that students are showing their work on problems 2-6 and then checking on the TI-Nspire or with you.
- It is suggested that students are not allowed to use CAS calculators on their upcoming assessments. CAS calculators are permitted on the AP Calculus Exam, but students need to gain an understanding of the basics since part of the AP Calculus AB Exam is non-calculator. Also, this ensures that students are assessed over their knowledge of the derivative formulas without solely relying on the calculator.
- Students will struggle more with finding the pattern for the quotient rule than the product rule, but the worksheet does provide some hints. Try to not to let students...
get too frustrated if they cannot determine the pattern. If several groups get it, have them explain their ideas to the class or to a struggling group.

- Another helpful hint for the students on the Quotient Rule table would be to distribute the terms in the numerator. The TI-Nspire factors out common terms, but that can make it trickier to find the pattern. There is space for students to do this in the table.

- On problem 12, students could use the Quotient Rule for \( \frac{1}{x^5} \) or they could rewrite it to be \( x^{-5} \) and then use the Power Rule. Both methods are correct, so it would be beneficial to have groups present the method that they used. Note to students that sometimes rewriting an equation first before taking the derivative can simplify a problem. In this case, either method yields about the same amount of work. It is important for students to remember that \( \frac{d}{dx}(1) = 0 \), which will be needed if students use the Quotient Rule. I find that students who use the Quotient Rule for derivatives frequently make errors because of this.

- On problem 13, students could either distribute first and then use just the Quotient Rule; or they could not distribute and use both the Product Rule and the Quotient Rule. Again, both methods are correct, so it would be beneficial to have groups present the method that they used. In this problem, it is much easier to first distribute and then use the Quotient Rule. Emphasize to students that if they can correctly simplify the function before using a derivative rule, then they should do so.

**Closure (about 5 minutes)**

- Have students answer the question, “What important points did you learn today?”
- Make sure students talk about what the product rule and quotient rule are. Also, make sure they discuss when they need to use those rules.
- Because the investigation will take most, if not all, of the class time, I recommend waiting until the next day to use the Quick Poll. At the beginning of the next day’s lesson, send students the Quick Poll (Product and Quotient Rules).
  - Have students complete this on their own.
  - Depending on how students do, go over the questions by showing the results and having a student explain the answer.
  - Note that the Quick Poll will accept any answers that are equivalent to the answer, so students do not need to simplify their answers.

**Homework:** Assign practice problems where students must apply the Power Rule, Product Rule, and Quotient Rule to find the derivative of a function, derivative at a point, equation of the tangent line, equation of the normal line, and solve application problems.
**Product and Quotient Rules (Student Document)**

**Product Rule**

1. Consider \( y = f(x)g(x) \) such that \( f(x) = \sin x \) and \( g(x) = 3x + 1 \). Then \( y = (3x + 1)\sin x \).

   a) Use your TI-Nspire calculator to find \( y' \).

   b) By hand, find \( f'(x) \) and \( g'(x) \).

Notice that while the derivative of the sum of two functions is sum of the two derivatives, the derivative of the product of two functions is NOT the product of their derivatives.

Now, we will try to discover the pattern for the product of two functions.

Complete the table below. Compute \( f'(x) \) and \( g'(x) \) by hand.

<table>
<thead>
<tr>
<th>( y )</th>
<th>( y' )</th>
<th>( f(x) )</th>
<th>( g(x) )</th>
<th>( f'(x) )</th>
<th>( g'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = (3x + 1)\sin x )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = (10x^3 + 2x)\sin x )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = 5x^2 \cos x )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = x^4 \cos x )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Determine an expression involving \( f(x) \), \( g(x) \), \( f'(x) \), and \( g'(x) \) that equals \( y' \). Use this pattern you found to write the Product Rule on the next page!
Product Rule:

\[ \frac{d}{dx} \left( f(x) g(x) \right) = \]

Also, \[ \frac{d}{dx}(uv) = \]

2) Test your conjecture for the Product Rule by creating a function that needs the Product Rule and test your rule. Make sure to write your original function and work out the derivative. Then check your answer by using the TI-Nspire CAS calculator.

For questions 3 and 4, use the Product Rule to find the derivative of the functions. Compute the derivatives by hand (show your work), and then check on the TI-Nspire.

3) \[ y = (x^2 + 1)(3x + 1) \]
4) \[ y = (2\sin x)(5x + 2) \]

5) Let \[ y = u \cdot v \] where \( u \) and \( v \) are functions in terms of \( x \) and have the values given below. Find \( y'(2) \).

\[
\begin{align*}
  u(2) &= 3 \\
  u'(2) &= -4 \\
  v(2) &= 1 \\
  v'(2) &= 2
\end{align*}
\]

6) A triangle currently has a base length of 5 cm and a height of 8 cm. The base is increasing at a rate of 2 cm/sec. The height is increasing at a rate of 3 cm/sec. How fast is the area of the triangle changing right now? Be sure to include units in your answer.
Quotient Rule

7. Consider \( y = \frac{f(x)}{g(x)} \) such that \( f(x) = 3x + 1 \) and \( g(x) = \sin x \). Then \( y = \frac{3x + 1}{\sin x} \).

a) Use your TI-Nspire calculator to find \( y' \).

b) By hand, find \( f'(x) \) and \( g'(x) \).

Notice that while the derivative of the difference of two functions is difference of the two derivatives, the derivative of the quotient of two functions is NOT the quotient of their derivatives.

Now, we will try to discover the pattern for the quotient of two functions. Complete the table below. Compute \( f'(x) \) and \( g'(x) \) by hand. In the third column, rewrite \( y' \) by expressing \( y' \) as a single fraction in which the terms in the numerator are distributed.
<table>
<thead>
<tr>
<th>( y' )</th>
<th>Rewrite ( y' )</th>
<th>( f(x) )</th>
<th>( g(x) )</th>
<th>( f'(x) )</th>
<th>( g'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = \frac{3x}{\sin x} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = \frac{x^3}{\sin x} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = \frac{\sin x}{x} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = \frac{5x^2}{\sin x} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Determine an expression involving $f(x)$, $g(x)$, $f'(x)$, and $g'(x)$ that equals $y'$. Use this pattern you found to write the Quotient Rule!

**Quotient Rule:**

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)$$

8) Test your conjecture for the Quotient Rule by creating a function that needs the Quotient Rule and test your rule. Write your original function and then work out the derivative. Then check your answer by using the TI-Nspire CAS calculator.

Now, test the Quotient Rule with the following functions with cosine.

9) \( y = \frac{5x^2}{\cos x} \)

10) \( y = \frac{x^4}{\cos x} \)

For questions 11-13, use the Product Rule and Quotient Rule along with previously learned rules where applicable to find the derivative of the functions. Compute the derivatives by hand (show your work), and then check on the TI-Nspire.

11) \( y = \frac{x^3 + 8}{4x + 5} \)

12) \( y = x^4 + \frac{1}{x^5} \)
13) \[ y = \frac{x(x^2 - 2x)}{\sin x} \]

Bonus: Write the formula for the derivative of \( \frac{g(x)}{f(x)} \).
**Product and Quotient Rules – TEACHER DOCUMENT**  

**Product Rule**

1. Consider \( y = f(x) g(x) \) such that \( f(x) = \sin x \) and \( g(x) = 3x + 1 \). Then \( y = (3x + 1) \sin x \).

   c) Use your TI-Nspire calculator to find \( y' \).

   \[
y' = (3x + 1) \cos x + 3 \sin x
   \]

   d) By hand, find \( f'(x) \) and \( g'(x) \).

   \[
f'(x) = 3 \quad \text{and} \quad g'(x) = \cos x
   \]

Notice that while the derivative of the sum of two functions is sum of the two derivatives, the derivative of the product of two functions is NOT the product of their derivatives.

Now, we will try to discover the pattern for the product of two functions.

Complete the table below. Compute \( f'(x) \) and \( g'(x) \) by hand.

<table>
<thead>
<tr>
<th>( y = (3x + 1) \sin x )</th>
<th>( y' )</th>
<th>( f(x) )</th>
<th>( g(x) )</th>
<th>( f'(x) )</th>
<th>( g'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (3x + 1) \cos x + 3 \sin x )</td>
<td>( 3x + 1 )</td>
<td>( \sin x )</td>
<td>3</td>
<td>( \cos x )</td>
<td></td>
</tr>
<tr>
<td>( (10x^3 + 2x) \sin x )</td>
<td>( 10x^3 + 2x )</td>
<td>( \sin x )</td>
<td>30x^2 + 2</td>
<td>( \cos x )</td>
<td></td>
</tr>
<tr>
<td>( 10x \cos x - 5x^2 \sin x )</td>
<td>( 5x^2 )</td>
<td>( \cos x )</td>
<td>10</td>
<td>( -\sin x )</td>
<td></td>
</tr>
<tr>
<td>( 4x^3 \cos x - x^4 \sin x )</td>
<td>( x^4 )</td>
<td>( \cos x )</td>
<td>4x^3</td>
<td>( -\sin x )</td>
<td></td>
</tr>
</tbody>
</table>

Determine an expression involving \( f(x) \), \( g(x) \), \( f'(x) \), and \( g'(x) \) that equals \( y' \). Use this pattern you found to write the Product Rule on the next page!
Product Rule:

\[
\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x)
\]

Also, \( \frac{d}{dx}(uv) = uv' + u'v \)

2) Test your conjecture for the Product Rule by creating a function that needs the Product Rule and test your rule. Make sure to write your original function and work out the derivative. Then check your answer by using the TI-Nspire CAS calculator.

Answers will vary.

For questions 3 and 4, use the Product Rule to find the derivative of the functions. Compute the derivatives by hand (show your work), and then check on the TI-Nspire.

3) \( y = (x^2 + 1)(3x + 1) \)

\[
y' = (x^2 + 1) \cdot 3 + 2x(3x + 1) \\
= 9x^2 + 2x + 3
\]

4) \( y = (2\sin x)(5x + 2) \)

\[
y' = (2\sin x) \cdot 5 + (2\cos x)(5x + 2) \\
= 10\sin(x) + 2\cos(x)(5x + 2)
\]

5) Let \( y = u \cdot v \) where \( u \) and \( v \) are functions in terms of \( x \) and have the values given below. Find \( y'(2) \).

\[
u(2) = 3 \\
u'(2) = -4 \\
v(2) = 1 \\
v'(2) = 2
\]

\[
y'(2) = 2
\]

6) A triangle currently has a base length of 5 cm and a height of 8 cm. The base is increasing at a rate of 2 cm/sec. The height is increasing at a rate of 3 cm/sec. How fast is the area of the triangle changing right now? Be sure to include units in your answer.

\[
A(t) = \frac{1}{2} \text{base} \cdot \text{height} \\
A'(t) = \frac{1}{2} \text{base} \cdot (\text{height})' + \frac{1}{2} (\text{base})' \cdot \text{height} \\
A'(0) = \frac{1}{2} (5)(3) + \frac{1}{2} (2)(8) \\
A'(0) = 15.5 cm^2 / \text{sec}
\]
Quotient Rule

7. Consider \( y = \frac{f(x)}{g(x)} \) such that \( f(x) = 3x + 1 \) and \( g(x) = \sin x \). Then \( y = \frac{3x + 1}{\sin x} \).

c) Use your TI-Nspire calculator to find \( y' \).

\[
y' = \frac{3\sin x - (3x + 1)\cos x}{\sin^2 x}
\]

d) By hand, find \( f'(x) \) and \( g'(x) \).

\[
f'(x) = 3 \text{ and } g'(x) = \cos x
\]

Notice that while the derivative of the difference of two functions is difference of the two derivatives, the derivative of the quotient of two functions is NOT the quotient of their derivatives.

Now, we will try to discover the pattern for the quotient of two functions. Complete the table below. Compute \( f'(x) \) and \( g'(x) \) by hand. In the third column, rewrite \( y' \) by expressing \( y' \) as a single fraction in which the terms in the numerator are distributed.
<table>
<thead>
<tr>
<th>( y = \frac{3x}{\sin x} )</th>
<th>( y' )</th>
<th>( \text{Rewrite } y' )</th>
<th>( f(x) )</th>
<th>( g(x) )</th>
<th>( f'(x) )</th>
<th>( g'(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{-3(x \cos x - \sin x)}{\sin^2 x} )</td>
<td>( \frac{-3x \cos x + 3 \sin x}{\sin^2 x} )</td>
<td>( 3x )</td>
<td>( \sin x )</td>
<td>( 3 )</td>
<td>( \cos x )</td>
<td></td>
</tr>
<tr>
<td>( y = \frac{x^3}{\sin x} )</td>
<td>( \frac{-x^3(x \cos x - 3 \sin x)}{\sin^2 x} )</td>
<td>( \frac{-x^3 \cos x + 3x^2 \sin x}{\sin^2 x} )</td>
<td>( x^3 )</td>
<td>( \sin x )</td>
<td>( 3x^2 )</td>
<td>( \cos x )</td>
</tr>
<tr>
<td>( y = \frac{\sin x}{x} )</td>
<td>( \frac{\cos x - \sin x}{x - x^2} )</td>
<td>( \frac{x \cos x - \sin x}{x^2} )</td>
<td>( \sin x )</td>
<td>( x )</td>
<td>( \cos x )</td>
<td>( 1 )</td>
</tr>
<tr>
<td>( y = \frac{5x^2}{\sin x} )</td>
<td>( \frac{-5x(x \cos x - 2 \sin x)}{\sin^2 x} )</td>
<td>( \frac{-5x^2 \cos x + 10x \sin x}{\sin^2 x} )</td>
<td>( 5x^2 )</td>
<td>( \sin x )</td>
<td>( 10x )</td>
<td>( \cos x )</td>
</tr>
</tbody>
</table>
Determine an expression involving $f(x)$, $g(x)$, $f'(x)$, and $g'(x)$ that equals $9y'$. Use this pattern you found to write the Quotient Rule!

**Quotient Rule:**

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

8) Test your conjecture for the Quotient Rule by creating a function that needs the Quotient Rule and test your rule. Write your original function and then work out the derivative. Then check your answer by using the TI-Nspire CAS calculator. Answers will vary.

Now, test the Quotient Rule with the following functions with cosine.

9) \[ y = \frac{5x^2}{\cos x} \]

\[ y' = \frac{10x \cos x + 5x^2 \sin x}{\cos^2 x} \]

10) \[ y = \frac{x^4}{\cos x} \]

\[ y' = \frac{4x^3 \cos x + x^4 \sin x}{\cos^2 x} \]

For questions 11-13, use the Product Rule and Quotient Rule along with previously learned rules where applicable to find the derivative of the functions. Compute the derivatives by hand (show your work), and then check on the TI-Nspire.

11) \[ y = \frac{x^3 + 8}{4x + 5} \]

\[ \frac{dy}{dx} = \frac{8x^3 + 15x^2 - 32}{(4x + 5)^2} \]

12) \[ y = x^4 + \frac{1}{x^5} \]

\[ \frac{dy}{dx} = 4x^3 - 5x^{-6} = 4x^3 - \frac{5}{x^6} \]

13) \[ y = \frac{x(x^2 - 2x)}{\sin x} \]

\[ \frac{dy}{dx} = \frac{(3x^2 - 4x)\sin x - (x^3 - 2x^2)\cos x}{\sin^2 x} = \frac{-x(x(2)\cos x - (3x - 4)\sin x)}{\sin^2 x} \]
Bonus: Write the formula for the derivative of $\frac{g(x)}{f(x)}$.

$$\frac{d}{dx}\left(\frac{g(x)}{f(x)}\right) = \frac{g'(x)f(x) - g(x)f'(x)}{(f(x))^2}$$

*Product and Quotient Rules Quick Polls Screenshots:*
Lesson 3: Discovering Velocity and Acceleration

Students were expected to discover the relationship between the position and velocity of a person’s movement given a graph of one of the functions. Overall, the lesson went well. Students were very interested in experimenting with the Calculator-Based Ranger (CBR) Motion Detectors. One group was a little slower than the rest to figure out how to match the graph, but they did not give up and eventually they were able to match it fairly well.

Overall the students were getting very competitive and really trying to be the best match. This level of excitement was great to see in a mathematics classroom because it does not happen too often. To encourage this friendly competitive nature, throughout the lesson, I put a screenshot of a group’s match on the board to allow other groups to see if they were doing better or should try it again. While this did make the lesson take longer, I think it was good for the students to see other groups’ work before the very end of the lesson.

I think this lesson helped to solidify their knowledge given a position graph. Students struggled more given the velocity graph. Because many of the groups were having a hard time distinguishing between the questions pertaining to acceleration versus speeding up or slowing down, I stopped the activity and had a whole class discussion about the differences between those two concepts. I think helped overall. I explained the acceleration only depends on the slope of the velocity graph. If the slope of the velocity graph is positive, then the person’s acceleration is positive. If the slope of the velocity graph is negative, then the person’s acceleration is negative. Students were able to grasp that idea easily.

The more complicated concept is determining if the person is slowing down or speeding up. A couple of students explained their thinking, which focused on the fact that if the velocity graph was moving away from the horizontal axis, then the person was speeding up. On the other hand, if the person was moving towards the horizontal axis, then the person was slowing down. This is a very helpful visual for many students because they can understand that the person’s velocity is either going away from a velocity of zero (speeding up) or approaching a velocity of zero (slowing down). I then added in the explanation about the signs of both the acceleration and velocity to determine if the person was speeding up or slowing down. If the signs for velocity and acceleration are the same, then the person is speeding up. If the signs for velocity and acceleration are not the same, then the person is slowing down.

While this lesson helped students to really understand position and velocity graphs, I believe another day would be needed on this material to completely solidify the material. The second day would need to include problems where students are given an equation for either the position or velocity and then asked questions. Having the first day where students have an opportunity to experiment and grasp the idea, should ensure that students can work with just equations as opposed to having a graph.
During the closure discussion, students brought up many valid points about different conceptual questions. Most groups were able to understand the idea that the position graph could have negative $y$-values, but we could not have that on a position graph that we were trying to match with a CBR motion detector. Groups were also able to discuss why the position graph with “corners” is not possible in the real world. I explained that the reason I chose a piece-wise graph made up of lines was that they could easily calculate the derivatives and graph the velocity function. Also, these are easy to reproduce with the CBR, which collects data at discrete intervals. Also, many of the AP Extended Response questions that deal with position and velocity graphs are in the format of piece-wise functions consisting of lines.

Collecting formative assessment data on what the students know after this lesson can determine how much students understood from the lesson. The Quick Poll allowed me to see if every student understood the difference between a position and a velocity graph. Most of my students were able to answer both of the questions correctly. There were still a few who were struggling, which is why I did another day on the topic to make sure everyone understood the material and could also answer questions given an equation instead of a graph.

This activity is modified from [1].
**Velocity and Acceleration – TEACHER DOCUMENT** (Activity Overview)

**Required Time:**
- 1 class (60 minutes)

**Standards**
- II. Derivatives
  - Applications of derivatives
  - Interpretation of the derivative as a rate of change in the context of velocity, speed, and acceleration.

**Objective**
- I can use derivatives to analyze straight line motion and solve other problems involving rates of change.

**Materials**
- **Student Resources:**
  - Velocity and Acceleration (Student Document): 1 per student
  - TI-Nspire CAS Calculators: 1 per group
    - Note this activity will work on a non-CAS TI-Nspire calculator.
  - TI-Nspire Velocity and Acceleration Document: uploaded on each of the TI-Nspire CAS Calculators that are in use
  - CBR Motion Detectors: 1 per group
  - Station for collecting data: 1 per group
    - Meter Sticks
    - Masking Tape
- **Teacher Resources:**
  - Velocity and Acceleration – Teacher Document
  - TI-Nspire Navigator Software
  - Velocity and Acceleration Quick Poll file

**Activity Overview (about 50 minutes)**
- Set up an activity station for each group before class (if teacher sets it up).
  - Ensure that each group has an area or have a maximum of two groups at each station. Also, include meter sticks taped on the ground or masking tape on the ground with meters marked for the data collection.
  - Note, students will need a minimum of six meters marked from position zero (at the wall/flat surface) to position 6 meters.
- Explain how the CBR Motion Detectors work to the whole class.
  - Connect the CBR to the calculator using the micro USB cable. Make sure the calculator is on. The CBR will turn on automatically. Data is collected through the calculator’s data collection application – e.g., page 1.3 of the preloaded activity document.
- The CBR motion detector bounces a signal off of a surface; it works best if it has a flat surface to collect data from. Students should hold the CBR motion detector, aim the detector at the flat surface/wall, and walk in a straight line to collect the data.

- When working with position and velocity graphs, the person can only move forward and backward (not side to side). When working with the CBR motion detector, *forward* means the distance between the CBR motion detector and the flat surface/wall is increasing while *backward* means the distance between the CBR motion detector and the flat surface/wall is decreasing. The person (and CBR) should face the same direction throughout the activity.

- Plug in the CBR motion detector into the TI-Nspire CAS calculator. Automatically, the data collection window will appear. Students need to open up the Velocity and Acceleration Document (“Home” → “2: My Documents” → “Velocity & Acceleration”) so they can see the graphs to match. To start collecting data in the data collection window, click on the green play button in the top left of the window. To stop, click the same spot, which now looks like a red stop sign. The data will be stored in the calculator until another set of data is collected in the same screen.

- Set up an activity station for each group (if students set up their own).
- Students work in groups of three to four to complete the *Velocity and Acceleration* (Student Document).
- Remind students to pay attention to the units for the graphs. The variable on the horizontal axis is *t* (for time), not *x*.
- Circulate around the room to ensure students are on task and are progressing on both the worksheet and with using the CBR Motion Detectors.
- Allow the students to experiment with the CBR Motion Detectors. Allow each member of the group to try to collect data and match the graphs.
- Make sure that students are answering the questions on the worksheet and explaining their answers. It is important for students to be able to communicate how they know their answers are correct.

**Closure (about 10 minutes)**

- Have students answer the question, “What important points did you learn today?”
- Make sure students discuss the differences and similarities of a position graph versus a velocity graph.
- Using the TI-Nspire Navigator Software, collect screenshots of each group’s graphs that they mapped out with the CBR Motion Detector. Show each of them on the board and determine which one is the closest match for questions 2 and 9. Make sure to discuss question 2(a). When showing the closest match for question 2, have a whole class discussion about how the position graph is not possible in the real world. Key words to include in the discussion are *differential* and *smooth curve*.
- Also, discuss whether it is possible for a position graph to have negative *y*-values. In question 1, the position graph does not go below the *t*-axis only because that
would be not possible to match using the CBR motion detectors. Make sure students understand that the y-values of a position graph can be negative.

- During the last 5 minutes, send students the Quick Poll (Velocity and Acceleration).
  - Have students complete this on their own.
  - If time allows, go over the questions by showing the results and having students explain the answers.

Homework: Assign practice problems where students are given different position and velocity graphs where they have to answer questions about the person’s movement. Later, include problems where students are just given equations for the position or velocity of a person and asked to answer questions.

*Velocity and Acceleration TI-Nspire Lesson Document Screenshots:*

1.1 – Title page

![Title page screenshot](image)

1.2 – Distance graph for question 1

![Distance graph screenshot](image)
1.3 – CBR Motion Detector data collection screen for question 2

2.1 – Velocity graph for question 7

3.1 – CBR Motion Detector data collection screen for question 9
Open the Velocity & Acceleration Document on your calculator: “Home” → “2: My Documents” → “Velocity & Acceleration”

*Note: When working with position and velocity graphs, the person can only move forward and backward (not side to side). When working with the CBR motion detector, forward means the distance between the CBR motion detector and the flat surface wall is increasing while backward means the distance between the CBR motion detector and the flat surface/wall is decreasing. The CBR measures distance between it and the wall it faces.

**Move to page 1.2 on the calculator.**
The graph is a position graph of a person. Answer the following questions based on the position graph and also explain what you are looking for on the graph to determine your answers.

1) When is the person…
   a) moving forward?
   b) moving backward?
   c) standing still?

**Move to page 1.3 on the calculator.**
2. Try to replicate the person’s movement from question 1 using the CBR detector. Hold the CBR in front of you while you walk directly toward or away from the wall. Make sure the CBR is always pointed at the wall. Is it possible to exactly match this position graph by walking? Explain completely.

3) What does the slope of the position graph tell us?
4) Now graph the slope of the position function (a.k.a. graph the derivative).

5) What is the name of the function you graphed in Question 4?

6) How are the position graph and the velocity graph related?

Move to page 2.1 on the calculator.
The velocity of a new person is now graphed. Answer the following questions based on the velocity graph and also explain what you are looking for on the graph to determine your answers.

7) When is the…
   a) person moving forward?
   b) person moving backward?
   c) person standing still?
   d) person’s acceleration positive?
   e) person’s acceleration negative?
   f) person’s acceleration zero?
   g) person speeding up?
   h) person slowing down?

Move to page 3.1 on the calculator.
8) Create your own position graph. Sketch your graph on the axes provided. The graph should not just be a straight line.

9) Use the CBR to try to replicate the position graph that you sketched in question 8. Check with me when finished! I will display the best replications for questions 2 and 9!
Open the Velocity & Acceleration Document on your calculator: “Home” → “2: My Documents” → “Velocity & Acceleration”

*Note:* When working with position and velocity graphs, the person can only move forward and backward (not side to side). When working with the CBR motion detector, forward means the distance between the CBR motion detector and the flat surface wall is increasing while backward means the distance between the CBR motion detector and the flat surface/wall is decreasing. The CBR measures distance between it and the wall it faces.

**Move to page 1.2 on the calculator.**
The graph is a position graph of a person. Answer the following questions based on the position graph and also explain what you are looking for on the graph to determine your answers.

1) When is the person…
   a) moving forward?
      \([0,3) \cup (8,9)\) because the slope of the line is positive.
   b) moving backward?
      \((3,5) \cup (7,8)\) because the slope of the line is negative.
   c) standing still?
      \((5,7) \cup (9,10]\) because the slope of the line is zero.

**Move to page 1.3 on the calculator.**
2. Try to replicate the person’s movement from question 1 using the CBR detector. Hold the CBR in front of you while you walk directly toward or away from the wall. Make sure the CBR is always pointed at the wall. Is it possible to exactly match this position graph by walking? Explain completely.
   It is not possible. The position graph is not differentiable, which makes the graph impossible in the real world. This is because one cannot instantaneously transition from moving forwards to moving backwards. In the real world, one must slow down and momentarily stop before changing direction. Thus, in the real world, this would need to be a differentiable (smooth) curve. Because the CBR collects data in discrete increments, it is theoretically possible to recreate this graph with the CBR, though.
3) What does the slope of the position graph tell us?
   The slope of the position graph tells us the velocity (speed and direction).

4) Now graph the slope of the position function (a.k.a. graph the derivative).

5) What is the name of the function you graphed in Question 4?
   This is the graph of the person’s velocity.

6) How are the position graph and the velocity graph related?
   The derivative of the position graph gives the velocity of the person.

Move to page 2.1 on the calculator.
The velocity of a new person is now graphed. Answer the following questions based on
the velocity graph and also explain what you are looking for on the graph to determine
your answers.

7) When is the...
   a) person moving forward?
      \((0,4.75) \cup (9,10]\) because the velocity is positive since the curve is above the \(t\)-
      axis.

   b) person moving backward?
      \((4.75,7)\) because the velocity is negative since the curve is below the \(t\)-axis.

   c) person standing still?
      \([7,9]\) because the velocity equals zero since the curve is on the \(t\)-axis. Note that
      students might mention \(t = 0\) and \(t = 4.75\). At \(t = 4.75\), the motion is
      transitioning from forwards to backwards. The velocity is 0, but the person isn’t
      really standing still. At \(t = 0\), it depends on whether the object was already
      moving before \(t = 0\), or it was at rest until \(t = 0\).
d) person’s acceleration positive?
\[ (0, 2) \cup (5, 7) \cup (9, 10) \] because the slope of the velocity curve is positive.

e) person’s acceleration negative?
\( (4,5) \) because the slope of the velocity curve is negative.

f) person’s acceleration zero?
\( (2,4) \cup (7,9) \) because the slope of the velocity curve is zero.

g) person speeding up?
\[ (0, 2) \cup (4.75,5) \cup (9,10) \] because the velocity graph moves away from the \( t \)-axis.

h) person slowing down?
\( (4,4.75) \cup (5,7) \) because the velocity graph moves toward the \( t \)-axis.

Move to page 3.1 on the calculator.
8) Create your own position graph. Sketch your graph on the axes provided. The graph should not just be a straight line.
   Answers will vary.

9) Use the CBR to try to replicate the position graph that you sketched in question 8.

Check with me when finished! I will display the best replications for questions 2 and 9!

Velocity and Acceleration Quick Polls Screenshots:
Lesson 4: Discovering the Chain Rule

Students were expected to discover the Chain Rule for derivatives and then work through some examples with their group. Overall, the lesson went well. I know students find the Chain Rule very difficult in general, but every group was able to at least explain the pattern to me in words. Some groups found it difficult to write out the pattern with correct notation, but several groups were even able to do that. While the examples were challenging to them, I think they were at an appropriate difficulty level.

In previous years, I used the more traditional classroom format where I reminded students about composite functions and then told them the Chain Rule. Over the years, I have found that students tend to struggle with when and how to use the Chain Rule. After exploring the Chain Rule with the TI-Nspire CAS calculators, I found that students were able to recognize when to use the Chain Rule more quickly than in previous years. Due to the drastic difference I saw this year versus previous years, I think experimenting with different composite functions for students to see the pattern is especially beneficial.

I chose the examples in the table very deliberately. I grouped similar functions together to help students see the pattern. Overall, students did very well with the completing the table. The sine functions were slightly more difficult for them to figure out, so I was happy that I put those at the end of the table. This allowed the students to get an idea of the pattern before including trigonometric functions. Students did very well with the first three sine functions: $y = \sin(2x)$, $y = \sin(x^2)$, and $y = \sin(x^3)$. They were able to break down the functions into their composite functions and see a pattern. The last two examples were much more difficult: $y = \sin^2(x)$ and $y = \sin^3(x)$. This is a common issue with trigonometric functions when raised to a power. For struggling groups, I recommended that students try to rewrite the original function to included parentheses. Another way to describe this to students would be to encourage students to write the function as one would to graph it in a calculator. After giving these hints, students were much more successful.

During this lesson, I ended up completing the last two examples (problems 6 and 7) on the board with the whole class. This allowed me to emphasize notation and some tips of rewriting the trigonometric functions to make it easier for them to determine the $f(x)$, $g(x)$, and $h(x)$ functions. I think it would be a good idea to speak to the entire class about notation because notation is extremely important. It also solidified exactly what the Chain Rule was to some students who were still struggling.

Collecting formative assessment data is also helpful to identify students who are struggling with the concept. The Quick Poll at the end of class was very helpful. If there is not enough time at the end of class, then using the Quick Poll at the beginning of the next class would suffice.
**Chain Rule – TEACHER DOCUMENT (Activity Overview)**

**Required Time:**
- 1 class (60 minutes)

**Standards**
- II. Derivatives
  - Computation of derivatives
  - Chain rule and implicit differentiation

**Objective**
- I can calculate derivatives using the chain rule in order to determine the derivative at a point and apply derivatives to application problems.

**Materials**
- Student Resources:
  - Chain Rule (Student Document): 1 per student
  - TI-Nspire CAS Calculators: 1 per student (or at least 1 per group)
- Teacher Resources:
  - Chain Rule – Teacher Document
  - TI-Nspire Navigator Software
  - Chain Rule Quick Poll file

**Activity Overview (about 50 minutes)**
- Students work in groups of three to four to complete the Chain Rule (Student Document).
- Circulate around the room to ensure students are on task and are progressing on the worksheet.
- Refrain from telling them the pattern for the Chain Rule. If students are struggling to see the pattern from the table, direct students to try to see a pattern that includes the Power Rule.
- Make sure that students are showing their work on all problems and then checking on the TI-Nspire. At this point, encourage students to show how the function can be broken down into its composite functions. After more practice, do not continue to make students show all the work, but it is helpful for today.
- It is suggested that students are not allowed to use CAS calculators on their upcoming assessments. CAS calculators are permitted on the AP Calculus Exam, but students need to gain an understanding of the basics since part of the AP Calculus Exam is non-calculator. Also, students may take subsequent courses in which they are not allowed to use CAS calculators.

**Closure (about 10 minutes)**
- Have students answer the question, “What important points did you learn today?”
- Make sure students talk about what the Chain Rule is and when and how to use it.
• During the last 5 minutes, send students the Quick Poll (Chain Rule).
  o Have students complete this on their own.
  o If time allows, go over the question by showing the results and having a
    student explain the answer.
  o Note that the Quick Poll will accept any answers that are equivalent to the
    answer, so students do not need to simplify their answers.

Homework: Assign practice problems where students must apply the Chain Rule to find
the derivative of functions. Wait to include application problems until students have a
solid foundation of the rule.
1. Consider \( y = (3x + 5)^4 \). Note that \( y = (3x + 5)^4 \) is a composite function because \( y = f(g(x)) \).
   
a) Find \( f(x) \) and \( g(x) \).

   b) By hand, find \( f'(x) \) and \( g'(x) \).

   c) Using your TI-Nspire calculator, find \( y' \).

Now, we will try to discover the pattern for the derivative of composite functions. Complete the table below. Each function is a composite function. Compute \( f'(x) \) and \( g'(x) \) by hand and then use your TI-Nspire calculator for \( y' \). Try to find the pattern! If you discover the pattern before you get to the end, try to find \( y' \) by hand and then check on the TI-Nspire.

<table>
<thead>
<tr>
<th>( y )</th>
<th>( f(x) )</th>
<th>( g(x) )</th>
<th>( f'(x) )</th>
<th>( g'(x) )</th>
<th>( y' )</th>
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<td>( y = \sin^3(x) )</td>
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</tbody>
</table>
Use the pattern you found to write the Chain Rule!

**Chain Rule:**

\[ \frac{d}{dx} \left( f \left( g(x) \right) \right) = \]

Note that you can have a composite function that is made up of more than two functions, but the same pattern holds.

2) Test your conjecture for the Chain Rule by creating a function that needs the Chain Rule and test your rule. Make sure to write your original function and find the derivative by hand. Then check your answer by using the TI-Nspire CAS calculator.

For questions 3-6, find the derivative of the functions. Compute the derivatives by hand (show your work), and then check on the TI-nspire.

3) \( y = (12x - 3)^4 \) 
4) \( y = \sin^6 x \)

5) \( y = 3\tan^5 (4x - 3) \) 
6) \( y = \left( 5 - \cos^3 (2x) \right)^9 \)

7) Find \( \frac{d^2 y}{dx^2} \) when \( y = \sec (3x) \).
Chain Rule – TEACHER DOCUMENT

Name:_________ KEY

1. Consider \( y = (3x + 5)^4 \). Note that \( y = (3x + 5)^4 \) is a composite function because \( y = f(g(x)) \).

   c) Find \( f(x) \) and \( g(x) \).
      \[ f(x) = x^4 \quad \text{and} \quad g(x) = 3x + 5 \]

   d) By hand, find \( f'(x) \) and \( g'(x) \).
      \[ f'(x) = 4x^3 \quad \text{and} \quad g'(x) = 3 \]

   c) Using your TI-Nspire calculator, find \( y' \).
      \[ y' = 12(3x + 5)^3 \]

Now, we will try to discover the pattern for the derivative of composite functions. Complete the table below. Each function is a composite function. Compute \( f'(x) \) and \( g'(x) \) by hand and then use your TI-Nspire calculator for \( y' \). Try to find the pattern! If you discover the pattern before you get to the end, try to find \( y' \) by hand and then check on the TI-Nspire.
<table>
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<th>( g(x) )</th>
<th>( f'(x) )</th>
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<td>(3x^2)</td>
<td>(3x^2 \cos(x^3))</td>
</tr>
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<td>(2x)</td>
<td>(\cos x)</td>
<td>(2 \sin x \cos x)</td>
</tr>
<tr>
<td>(\sin^3(x))</td>
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<td>(\sin x)</td>
<td>(3x^2)</td>
<td>(\cos x)</td>
<td>(3 \sin^2 x \cos x)</td>
</tr>
</tbody>
</table>

Use the pattern you found to write the Chain Rule!

**Chain Rule:**

\[
\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)
\]

Note that you can have a composite function that is made up of more than two functions, but the same pattern holds.

2) Test your conjecture for the Chain Rule by creating a function that needs the Chain Rule and test your rule. Make sure to write your original function and find the derivative by hand. Then check your answer by using the TI-Nspire CAS calculator.

*Answers will vary.*
For questions 3-6, find the derivative of the functions. Compute the derivatives by hand (show your work), and then check on the TI-nspire.

3) \( y = (12x - 3)^4 \)  
   \[ y' = 48(12x - 3)^3 \]

4) \( y = \sin^6 x \)  
   \[ y' = 6\sin^5 x\cos x \]

5) \( y = 3\tan^5 (4x - 3) \)  
   \[ y' = 60\tan^4 (4x - 3)\sec^2 (4x - 3) \]

6) \( y = (5 - \cos^3 (2x))^9 \)  
   \[ y' = 54(5 - \cos^3 (2x))^8 \cos^2 (2x)\sin (2x) \]

7) Find \( \frac{d^2y}{dx^2} \) when \( y = \sec (3x) \).
   \[ y' = 3\sec (3x)\tan (3x) \]
   \[ y'' = 9\sec (3x)\tan^2 (3x) + 9\sec^3 (3x) \]

*Chain Rule Quick Polls Screenshots:*
Lesson 5: Discovering the Mean Value Theorem

Students were expected to discover what the Mean Value Theorem is and when it can be applied. Overall, the lesson went well. It proved to be a little challenging with the technology piece, but the lesson went better than in previous years without technology. The students were engaged in the lesson and most were trying hard to get the tangent line to have exactly the same slope as the secant line.

I had developed an activity for the Mean Value Theorem previously, but there was no technology piece involved. Instead, I would give the students graphs and have them draw in the secant and possible tangent lines with a ruler. Using the technology piece this year greatly helped for students to see how the tangent line changes along the curve. It sparked many rich conversations about what happens to the slope at a cusp from problem on page 2.1. Students could better see and understand that the slope of the tangent line approaches negative infinity on one side and positive infinity on the other side of the cusp.

Note that on page 4.1, the function is a piecewise function. In order to ensure students understood where the function was defined at \( x = 1 \), I included the equation on the student document. This helped to clarify the graph on the TI-Nspire since it does not show the open circle and the closed circle at \( x = 1 \). This was important to disclose to students, so that they understand that the function is not continuous on \([a, b]\) but is differentiable on \((a, b)\). Most students were able to understand that. This helps to solidify the necessary conditions of the Mean Value Theorem: the function must be continuous on \([a, b]\) and differentiable on \((a, b)\).

Students were able to successfully analyze the data that they collected from the four examples and answer questions 14 through 16 well with their groups. Some groups struggled with determining exactly what I wanted in problems 17 through 20. Once I gave them a little more direction, they were able to answer the questions easily. They mostly just wanted to have an equation because they thought the answers were too obvious.

I found that the Closure part of this lesson was extremely important. It helped to have a whole class discussion to explain the importance of the Mean Value Theorem. It also helped to discuss notation and the fact that the theorem uses \( f'(c) \) not \( f'(x) \). The Quick Poll was again very helpful in collecting formative assessment data. I took another day to work with the Mean Value Theorem following the TI-Nspire activity. From the Quick Poll, I could tell that everyone knew the theorem, but some still needed help in applying it. The following day consisted of examples using equations, rather than graphs, and also working with some application problems.
**Mean Value Theorem – TEACHER DOCUMENT (Activity Overview)**

Required Time:
- 1 class (60 minutes)

Standards
- II. Derivatives
  - Derivative as a function
  - The Mean Value Theorem and its geometric interpretation

Objective
- I can determine if the Mean Value Theorem applies and if so, I can estimate the $x$-values where the tangent line is parallel to the secant line.

Materials
- Student Resources:
  - Mean Value Theorem (Student Document): 1 per student
  - TI-Nspire CAS Calculators: 1 per student (or at least 1 per group)
  - TI-Nspire Mean Value Theorem Document: uploaded on each of the TI-Nspire CAS Calculators that are in use
- Teacher Resources:
  - Mean Value Theorem – Teacher Document
  - TI-Nspire Navigator Software
  - Mean Value Theorem Quick Poll file

Activity Overview (about 50 minutes)
- Students work in groups of three to four to complete the *Mean Value Theorem (Student Document)*.
- Circulate around the room to ensure students are on task and are progressing on the worksheet and through the TI-Nspire Document.
- If students are struggling with using the calculators, you could address the whole class and demonstrate how to grab the point of tangency and drag it around to change the tangent line. If students are progressing well, let them experiment to determine how to use the technology on their own.
- It is not important for students to get the tangent line’s slope to be exactly equal to the slope of the secant line. With the technology involved, it can be difficult to get the point of tangency to move in small enough increments in order to get the slopes to be equal. Allow close answers to be correct. The important idea is that in theory the slopes should be equal. If students really want to get exactly the same slopes, zooming in can help to move the point of tangency in smaller increments.
- Make sure students are understanding the vocabulary: continuous and differentiable.
- Also, make sure students are clear about the interval notation and the differences between round parentheses and square brackets.
Closure (about 10 minutes)

- Have students answer the question, “What important points did you learn today?”
- Make sure students talk about what has to be true in order for the Mean Value Theorem to be applied along with the result of the Mean Value Theorem, in both the graphical aspect and the formula.
- Introduce the fact that in the Mean Value Theorem, it is customary that students use a $c$ instead of an $x$ when solving for the $x$-coordinate where the slope of the tangent line is equal to the slope of the secant line. This is due to the fact that the theorem gives a result about a specific point $c$, not a variable point $x$.
  - During the last 5 minutes, send students the Quick Poll (Mean Value Theorem).
  - Have students complete this on their own.
  - If time allows, go over the question by showing the results and having a student explain the answer.

Homework: Assign practice problems where students must first determine if the Mean Value Theorem can be applied. Then in the problems where the Mean Value Theorem does apply, have the students solve for the $c$-value.

*Mean Value Theorem TI-Nspire Lesson Document Screenshots:*

1.1 – Title page
1.2 – Graph for questions 1-4

2.1 – Graph for questions 5-7

3.1 – Graph for questions 8-10
4.1 – Graph for questions 11-13
Open the Mean Value Theorem Document on your calculator: “Home” → “2: My Documents” → “Mean Value Theorem”

Move to page 1.2 on the calculator.
In each problem, a function is graphed (thin blue) with a secant line drawn from $x = a$ to $x = b$ (dashed black) and a tangent at some point in $(a, b)$ (thick green). The equations are shown for the secant line (black) and the tangent line (green). The coordinates of the point of tangency are also listed (black).

1) Is the function continuous on $[a, b]$?

2) Is the function differentiable on $(a, b)$?

3) When lines are parallel, what is true about their slopes?

4) Grab the point of tangency and move it to try to find a tangent line that is parallel to the secant line drawn from $x = a$ to $x = b$. Were there any points where the tangent line was parallel to the secant line? If yes, how many of these parallel tangent lines were there, and at what $x$-coordinates did these occur?

Move to page 2.1 on the calculator.

5) Is the function continuous on $[a, b]$?

6) Is the function differentiable on $(a, b)$?

7) Grab the point of tangency and move it to try to find a tangent line that is parallel to the secant line drawn from $x = a$ to $x = b$. Were there any points where the tangent line was parallel to the secant line? If yes, how many of these parallel tangent lines were there, and at what $x$-coordinates did these occur?
Move to page 3.1 on the calculator.

8) Is the function continuous on \([a, b]\)?

9) Is the function differentiable on \((a, b)\)?

10) Grab the point of tangency and move it to try to find a tangent line that is parallel to the secant line drawn from \(x = a\) to \(x = b\). Were there any points where the tangent line was parallel to the secant line? If yes, how many of these parallel tangent lines were there, and at what \(x\)-coordinates did these occur?

Move to page 4.1 on the calculator.

Note: The function shown in the graph is given by the following rule:

\[
f(x) = \begin{cases} 
(x-1)^2 + 1, & x < 1 \\
5, & x \geq 1
\end{cases}
\]

Also, note that \(a = 0\) and \(b = 1\).

11) Is the function continuous on \([a, b]\)?

12) Is the function differentiable on \((a, b)\)?

13) Grab the point of tangency and move it to try to find a tangent line that is parallel to the secant line drawn from \(x = a\) to \(x = b\). Were there any points where the tangent line was parallel to the secant line? If yes, how many of these parallel tangent lines were there, and at what \(x\)-coordinates did these occur?

Look back at your previous work to answer the following questions:

14) If a function is continuous on the closed interval \([a, b]\), is there always a tangent line that is parallel to the secant line from \(x = a\) and \(x = b\)? Explain your answer.

15) If a function is differentiable on the open interval \((a, b)\), is there always a tangent line that is parallel to the secant line? Explain your answer.
16) For there to always be at least one tangent line parallel to the secant line from $x = a$ to $x = b$, the function must be ________________ on the closed interval $[a, b]$ and ________________ on the open interval $(a, b)$.

The statement in question 16 is part of the Mean Value Theorem. The complete theorem includes an equation to find the $x$-coordinate of the point of tangency where the tangent line is parallel to the secant line. Below you will build the equation:

17) What is the slope of the tangent line of a function $f(x)$ at $x = c$?

18) State the formula for the slope of the secant line of a function $f(x)$ over the interval $[a, b]$:

19) The Mean Value Theorem describes the point(s) where the slopes of the tangent line and secant line are equal to each other, so set your expressions from questions 17 and 18 equal.
Open the Mean Value Theorem Document on your calculator: “Home” → “2: My Documents” → “Mean Value Theorem”

Move to page 1.2 on the calculator.
In each problem, a function is graphed (thin blue) with a secant line drawn from \( x = a \) to \( x = b \) (dashed black) and a tangent at some point in \((a, b)\) (thick green). The equations are shown for the secant line (black) and the tangent line (green). The coordinates of the point of tangency are also listed (black).

1) Is the function continuous on \([a, b]\)?
   Yes
2) Is the function differentiable on \((a, b)\)?
   Yes
3) When lines are parallel, what is true about their slopes?
   Parallel lines have the same slope.
4) Grab the point of tangency and move it to try to find a tangent line that is parallel to the secant line drawn from \( x = a \) to \( x = b \). Were there any points where the tangent line was parallel to the secant line? If yes, how many of these parallel tangent lines were there, and at what \( x \)-coordinates did these occur?
   Yes, there is one tangent line that is parallel to the secant line.

Move to page 2.1 on the calculator.
5) Is the function continuous on \([a, b]\)?
   Yes
6) Is the function differentiable on \((a, b)\)?
7) Grab the point of tangency and move it to try to find a tangent line that is parallel to the secant line drawn from $x = a$ to $x = b$. Were there any points where the tangent line was parallel to the secant line? If yes, how many of these parallel tangent lines were there, and at what $x$-coordinates did these occur?

No there are no tangent lines that are parallel to the secant line.

Move to page 3.1 on the calculator.

8) Is the function continuous on $[a, b]$?

Yes

9) Is the function differentiable on $(a, b)$?

Yes

10) Grab the point of tangency and move it to try to find a tangent line that is parallel to the secant line drawn from $x = a$ to $x = b$. Were there any points where the tangent line was parallel to the secant line? If yes, how many of these parallel tangent lines were there, and at what $x$-coordinates did these occur?

Yes, there are two tangent lines that are parallel to the secant line.

Move to page 4.1 on the calculator.

Note: The function shown in the graph is given by the following rule:

$$f(x) = \begin{cases} \frac{(x-1)^2}{5} + 1, & x < 1 \\ 5, & x \geq 1 \end{cases}$$

Also, note that $a = 0$ and $b = 1$.

11) Is the function continuous on $[a, b]$?

No
12) Is the function differentiable on \((a,b)\)?

Yes

13) Grab the point of tangency and move it to try to find a tangent line that is parallel to the secant line drawn from \(x=a\) to \(x=b\). Were there any points where the tangent line was parallel to the secant line? If yes, how many of these parallel tangent lines were there, and at what \(x\)-coordinates did these occur?

No there are no tangent lines that are parallel to the secant line.

Look back at your previous work to answer the following questions:

14) If a function is continuous on the closed interval \([a,b]\), is there always a tangent line that is parallel to the secant line from \(x=a\) and \(x=b\)? Explain your answer.

Not every time. The function on page 2.1 is an example of a continuous function that does not have a tangent line that is parallel to the secant line.

15) If a function is differentiable on the open interval \((a,b)\), is there always a tangent line that is parallel to the secant line? Explain your answer.

Not every time. The function on page 4.1 is an example of a differentiable function that does not have a tangent line that is parallel to a secant line.

16) For there to always be at least one tangent line parallel to the secant line from \(x=a\) to \(x=b\), the function must be \_________continuous\_________ on the closed interval \([a,b]\) and \_________differentiable\_________ on the open interval \((a,b)\).

The statement in question 16 is part of the Mean Value Theorem. The complete theorem includes an equation to find the \(x\)-coordinate of the point of tangency where the tangent line is parallel to the secant line. Below you will build the equation:

17) What is the slope of the tangent line of a function \(f(x)\) at \(x=c\):

\[ f'(c) \]

18) State the formula for the slope of the secant line of a function \(f(x)\) over the interval \([a,b]\):

\[ \frac{f(b) - f(a)}{b - a} \]

19) The Mean Value Theorem describes the point(s) where the slopes of the tangent line and secant line are equal to each other, so set your expressions from questions 17 and 18 equal.
\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

**Mean Value Theorem TI-Nspire Quick Poll Screenshots:**

What is the equation from the Mean Value Theorem that allows you to calculate the \(x\)-coordinate \(c\) where the tangent line is parallel to the secant line?

- \(f'(c) = \frac{f(b) - f(a)}{b - a}\)
- \(f'(c) = \frac{f(b) - f(a)}{b - a}\)

Find the value of \(c\) in the interval \((a,b)\) that satisfies the Mean Value Theorem for the following function:

\(f(x) = x^2 + 2x - 1\) on \([0,4]\)
Lesson 6: Discovering Slope Fields

Students were expected to discover what a slope field is and how it can help one find the particular solution to a differential equation given an initial value. Overall, the students were able to use the differential equations to graph their own slope fields and calculate the exact slopes given a differential equation. Students were able to find the particular solution in problem 7 but greatly struggled with problem 10. This was due to the fact that in question 10, students had to first separate the variables before taking the antiderivative of both sides. One of the antiderivatives in that problem also involved \( \ln|y| \), so it was more difficult for the students to solve for \( y \). It might have been a good idea to work though problem 10 as a whole class, but I did not do that. I just went around to each group and worked through it with them. Some groups were able to get far in the process, but had a hard time determining the constant.

The students, though, were very involved throughout the entire lesson. Students found it intriguing that they could move the initial value on the slope fields and see the changes that would happen to the equation. Due to this process being very interactive, I feel that these students understand slope fields better than my students in previous years.

Using the Quick Poll to gather some formative assessment data, I could tell that all of my students were able to find the slope of the function given the differential equation and a point. Most, but not all, of my students were able to solve for the particular solution given a differential equation that did not involve separating the variables. Due to this data, the following day I had my students do more practice problems, solving differential equations where variables did and did not have to be separated. This allowed the students who were still struggling with that process to have more time to understand and grasp the concept. This also allowed me to have time to talk about the notation that needs to be involved when dealing with an absolute value in the equation.
**Slope Fields – TEACHER DOCUMENT (Activity Overview)**

**Required Time:**
- 1 class (60 minutes)

**Standards**
- II. Derivatives
  - Applications of derivatives

**Objective**
- I can interpret and create slope fields and use them to graph a solution.

**Materials**
- **Student Resources:**
  - Slope Fields (Student Document): 1 per student
  - TI-Nspire CAS Calculators: 1 per student (or at least 1 per group)
  - TI-Nspire Slope Fields Document: uploaded on each of the TI-Nspire CAS Calculators that are in use
- **Teacher Resources:**
  - Slope Fields – Teacher Document
  - TI-Nspire Navigator Software
  - Slope Fields Quick Poll file

**Activity Overview (about 50 minutes)**
- Students work in groups of three to four to complete the *Slope Fields (Student Document).*
- Circulate around the room to ensure students are on task and are progressing on the worksheet and through the TI-Nspire Document.
- If students are struggling with using the calculators, you could address the whole class and demonstrate how to grab the point and drag it around the screen to change the graph of the function. If students are progressing well, let them experiment to determine how to use the technology on their own.
- Make sure students are understanding that the differential equation that is given is different from the equation $y = f(x)$. Once students start drawing the particular solution and finding the function, they must understand that it is $y = f(x)$ and no longer the differential equation.
- When students solve differential equations, ensure that they first separate the variables before antidifferentiating. Not doing so is a very common error and if this error is made, the students cannot earn any points for the rest of the problem according to the Advanced Placement Scoring Guidelines.
Closure (about 10 minutes)

- Have students answer the question, “What important points did you learn today?”
- Make sure students discuss the differences between the differential equation and the particular solution. When students solve a differential equation, again emphasize that the variables must be separated first.
- During the last 5 minutes, send students the Quick Poll (Slope Fields).
  - Have students complete this on their own.
  - If time allows, go over the question by showing the results and having a student explain the answer.

Homework: Assign practice problems where students must calculate the slopes at several points, given the differential equation. Include problems where students graph a slope field with a specified number of points. Also, have students graph a particular solution given an initial value. Students could also be expected to solve the differential equation algebraically.

Slope Fields TI-Nspire Lesson Document Screenshots:

1.1 – Title page
1.2 – Slope field for questions 2-4

2.1 – Slope field for question 5

2.2 – Slope field for questions 6 and 7
3.1 – Differential equation for question 8

\[ \frac{dy}{dx} = -xy \]

3.2 – Slope field for questions 9 and 10

3.3. – Graph for question 11
An equation involving a derivative is called a differential equation.

1) Find the general solution to the differential equation \( \frac{dy}{dx} = \sec^2 x + e^x + \frac{1}{x} \). In other words, solve for \( y \). (Your answer will be a function.)

Now check your answers on your TI-Nspire. To do so from the home screen, go to “Calculate” → “Menu” → “4: Calculus” → “3: Integral”. Note: The integral will appear as a definite integral with places to type in the limits. If you leave the limits blank, though, the calculator realizes that it is an indefinite integral.

Derivatives tell us the __________ of a function at any point on the curve.

We can use the derivative of a function to draw a small piece of the curve at a point based on its slope, which approximates the solution curve that passes through that point. Repeating the process at many different points in the plane yields a slope field.


Move to page 1.2 on the calculator.

This shows the slope field for the differential equation \( \frac{dy}{dx} = -\frac{x}{y} \).

2) When is the slope equal to zero?

   a) How can you tell from the differential equation?

   b) How can you tell based on the slope field?

3) When is the slope undefined?

   a) How can you tell from the differential equation?

   b) How can you tell based on the slope field?
4) Verify that the slopes in the slope field match the value of the derivative at the following points for the differential equation \( \frac{dy}{dx} = -\frac{x}{y} \):

   a) \( \left. \frac{dy}{dx} \right|_{(1,1)} = \) 

   b) \( \left. \frac{dy}{dx} \right|_{(3,1)} = \) 

   c) \( \left. \frac{dy}{dx} \right|_{(-1,4)} = \)

Move to page 2.1 on the calculator.

This shows the slope field for the differential equation \( \frac{dy}{dx} = \cos(x) \), which is also shown below.

5) Sketch a possible solution \( y = f(x) \) for this differential equation with the initial condition \( f(\pi) = 3 \).

Move to page 2.2 on the calculator.

Check your solution from above with the graph on the calculator. Adjust your sketch as needed. Now, grab the point \((\pi,3)\) on the graph and drag it to see other graphs of \( y = f(x) \) that have a different initial value.

6) What family of functions is the equation \( y = f(x) \)?

7) Use calculus to find the particular solution \( y = f(x) \) with the initial condition \( f(\pi) = 3 \).
Move to page 3.1 on the calculator.

8) Construct a slope field for the differential equation \( \frac{dy}{dx} = -xy \) through the fifteen points shown.

Move to page 3.2 on the calculator.

This shows the slope field for the differential equation \( \frac{dy}{dx} = -xy \). Check your graph above to see if the slopes of the line segments match.

9) On your slope field from question 8, sketch a possible graph for the particular solution \( y = f(x) \) to the differential equation with the initial condition \( f(0) = 3 \).

10) Use calculus to find the particular solution \( y = f(x) \) to the differential equation \( \frac{dy}{dx} = -xy \) with the initial condition \( f(0) = 3 \).

Now, have the calculator graph the solution. Bring up the equation menu by clicking “Tab.” Go up to “y1” and where you see “(x0,y10):” input “(0,3)”. Then click “Enter.” Check that the result matches your sketch.

Move to page 3.3 on the calculator.

11) Graph your equation from question 10. Bring up the equation menu by clicking “Tab.” Type in your equation from question 10 and graph it. Check to make sure the graphs are similar! Note that the further you are from the initial condition, the more the graphs might differ. If they differ greatly, check with me.
**Slope Fields** – **TEACHER DOCUMENT**

Name: ________________

An equation involving a derivative is called a **differential equation**.

1) Find the general solution to the differential equation $\frac{dy}{dx} = \sec^2 x + e^x + \frac{1}{x}$. In other words, solve for $y$. (Your answer will be a function.)

   $y = \tan x + e^x + \ln |x| + C$

Now check your answers on your TI-Nspire. To do so from the home screen, go to “Calculate” → “Menu” → “4: Calculus” → “3: Integral”. Note: The integral will appear as a definite integral with places to type in the limits. If you leave the limits blank, though, the calculator realizes that it is an indefinite integral.

Derivatives tell us the ___slope________ of a function at any point on the curve.

We can use the derivative of a function to draw a small piece of the curve at a point based on its slope, which approximates the solution curve that passes through that point. Repeating the process at many different points in the plane yields a **slope field**.

**Slope Fields Document**: “Home” → “2: My Documents” → “Slope Fields”

**Move to page 1.2 on the calculator.**

This shows the slope field for the differential equation $\frac{dy}{dx} = -\frac{x}{y}$.

2) When is the slope equal to zero?

   The slope equals zero when $x = 0$ and $y \neq 0$

   OR on the y-axis, but not at the origin.

   a) How can you tell from the differential equation?

      Algebraically, the derivative equals zero when the numerator equals zero and the denominator does not equal zero.

   b) How can you tell based on the slope field?

      Graphically, the slope equals zero when the line segments are horizontal.

3) When is the slope undefined?

   The slope is undefined when $y = 0$

   OR on the x-axis.

   a) How can you tell from the differential equation?

      Algebraically, the derivative is undefined when the denominator equals zero.
b) How can you tell based on the slope field?
   Graphically, slope is undefined when the line segments are vertical.

4) Verify that the slopes in the slope field match the value of the derivative at the following points for the differential equation \( \frac{dy}{dx} = -\frac{x}{y} \):

   a) \( \frac{dy}{dx} \bigg|_{(1,1)} = -1 \)
   b) \( \frac{dy}{dx} \bigg|_{(3,1)} = -3 \)
   c) \( \frac{dy}{dx} \bigg|_{(-1,4)} = \frac{1}{4} \)

Move to page 2.1 on the calculator.

This shows the slope field for the differential equation \( \frac{dy}{dx} = \cos(x) \), which is also shown below.

5) Sketch a possible solution \( y = f(x) \) for this differential equation with the initial condition \( f(\pi) = 3 \).

Move to page 2.2 on the calculator.

Check your solution from above with the graph on the calculator. Adjust your sketch as needed. Now, grab the point \((\pi, 3)\) on the graph and drag it to see other graphs of \( y = f(x) \) that have a different initial value.

6) What family of functions is the equation \( y = f(x) \)?
   Sine family of functions
   OR \( \sin x + C \)

7) Use calculus to find the particular solution \( y = f(x) \) with the initial condition \( f(\pi) = 3 \).
   \( y = \sin x + 3 \)
Move to page 3.1 on the calculator.

8) Construct a slope field for the differential equation \( \frac{dy}{dx} = -xy \) through the fifteen points shown.

![Slope Field Image]

Move to page 3.2 on the calculator.

This shows the slope field for the differential equation \( \frac{dy}{dx} = -xy \). Check your graph above to see if the slopes of the line segments match.

9) On your slope field from question 8, sketch a possible graph for the particular solution \( y = f(x) \) to the differential equation with the initial condition \( f(0) = 3 \).

![Graph Image]

10) Use calculus to find the particular solution \( y = f(x) \) to the differential equation \( \frac{dy}{dx} = -xy \) with the initial condition \( f(0) = 3 \).

\[ y = 3e^{-\frac{1}{2}x^2} \]
Now, have the calculator graph the solution. Bring up the equation menu by clicking “Tab.” Go up to “y1” and where you see “(x₀,y₁₀):” input “(0,3).” Then click “Enter.” Check that the result matches your sketch.

**Move to page 3.3 on the calculator.**
11) Graph your equation from question 10. Bring up the equation menu by clicking “Tab.” Type in your equation from question 10 and graph it. Check to make sure the graphs are similar! Note that the further you are from the initial condition, the more the graphs might differ. If they differ greatly, check with me.

*Slope Fields TI-Nspire Quick Poll Screenshots:*

![Slope Fields TI-Nspire Quick Poll Screenshots](image)
Conclusion

Overall, I believe these discovery lessons enhance student learning. Throughout the six lessons that I used during the 2016-2017 school year, I saw students who were engaged and motivated to learn. These students enjoyed discovering the new math concepts. I saw excitement and inquiries that did not happen in my previous traditional classroom setting. I believe that more teachers should be moving away from the traditional classroom and trying to incorporate these types of discovery lessons. While not every mathematical concept caters to a discovery lesson, I believe that many do. Some other topics that I might develop in the future would include Riemann Sums and the Average Value of a Function.

A definite drawback of these discovery lessons is that they are time consuming. Each of these lessons takes an entire period of about 60 minutes. Even with the entire class period, more time is needed to practice the mathematical concepts. These lessons help to ensure students have a deeper understanding of the concept, which takes time. In the first day of discovering the concepts, they do not have the repetition in practice problems that is need to ensure mastery. Therefore, more time is needed to practice but in the end it does lead to better understanding.

While being time consuming is one drawback to these types of discovery lessons, I believe that the benefits greatly exceed this one negative. Students leave with a deeper understanding of the material, and the activity provides an opportunity for students to be engaged in exploring mathematical concepts. These activities can help foster an inquiring mind that leads to better problem solvers and researchers for the future. Teaching is not just about getting students to learn facts but helping them become lifelong learners where they question concepts and investigate to see if there is a better solution.
References
