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AREA AND VOLUME WHERE DO THE FORMULAS COME FROM?

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WHERE DO THE FORMULAS COME FROM?

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This essay of Roger L. Yarnell is hereby accepted:

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I certify that this is the original document

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Author – Roger L. Yarnell Date

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1. INTRODUCTION

What are area and volume? This was a question that was posed to incoming geometry students that have previously worked with these measurements in middle school. There were a variety of responses to the question, how are area and volume defined? Approximately half of the students had a general idea of what area and volume measure. The students stated that area is the amount of space in a two dimensional figure and volume is the amount of space in a three dimensional object, the students made no mention as to the units used to measure area and volume. Only one student stated that area is measured in square units and volume in cubic units. Other student responses for area, included multiplying two sides (the length times the width) and the length around an object. Yet other student responses for volume, included the width and length combined, the mass of an object, the weight of an object, how much area is in an object, and it is found by multiplying length, width and height together.

Why are there so many different misconceptions for area and volume? Part of the reason for this may be that the students are given the formulas and are asked to simply compute the area and volume of different shapes and objects by substituting in values and plugging it into a calculator to get an answer. This does not give any fundamental understanding as to what the students are calculating or give them the meaning as to how this formula works for the selected object. As a result, students do not fully understand what their answer represents nor does it have any real or true meaning.

The students need to know the meaning of area and volume and where the concepts originate in order to give them meaning. When the students understand where concepts come from they can obtain a deeper understanding and this will allow them to know the meaning of their solution. The students have been shown why the area of a rectangle is length times width but not for other shapes. This is why the students remember and think area is simply length times width. The students use the other formulas not knowing why those formulas work to find area for a particular figure.

As the instructor, I was not proving all the formulas for area and volume of objects. I would prove some of the formulas, such as area of a rectangle, square, parallelogram, triangle, and trapezoid and the volume of a rectangle prism. I thought these could be done with complete understanding but not the other formulas with more complicated proofs. Due to time constraints, the decision was made that it was more important for the students to apply the formulas to problem solving. Problem solving is a very important skill, but after further education in the graduate courses, it was revealed how important it is for the students to understand where the formulas for area and volume originate and how exactly the formulas give the desired result.

This is a unit that includes the history of area and volume and will demonstrate and prove the formulas of each figure and shape that are used to calculate both area and volume. This unit will have the student gain an understanding of the mathematics behind both area and volume. This unit will also further enhance an appreciation and memory of what the students are calculating and understand the meaningfulness of their answer.

What are area and volume? They are defined as follows [6]:

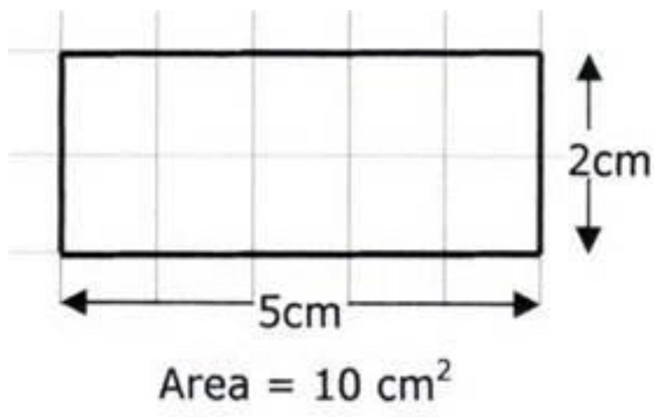
AREA is the number of unit square inside a closed region.

VOLUME is the number of unit cubes in a solid figure.

The earliest beginnings of geometry can be traced back to around 3000 BC to ancient Egypt and Mesopotamia. They had used lengths, angles, areas, and volumes for construction and surveying purposes. [5] The Babylonian and Egyptian people knew the basic formulas for area and volume. They had descriptions for the area of several figures, some correct and for others, such as the triangle, we are not sure they were completely correct. They regarded geometry as a practical tool for their civilization. It is believed that they did not have deductive proofs of their formulas. [7]

2. BASIC AREA FORMULAS

The area of a rectangle is probably the easiest to visualize. When the rectangle is divided into unit squares the formula for area can be developed. By counting these unit squares, the area of the rectangle can be determined. Also, the value for area can be found by doing repeated addition. Count the number of squares in each row, the number of rows within the rectangle, and the product will be the area. As in the diagram, there are five squares in a row and this is to be counted two times because there are two rows. Repeated addition in this manner is multiplication, so the number of squares can be found by multiplying the dimensions of length and width as this method of counting is a definition of multiplication. Therefore, the area of a rectangle can be defined as length multiplied by width.

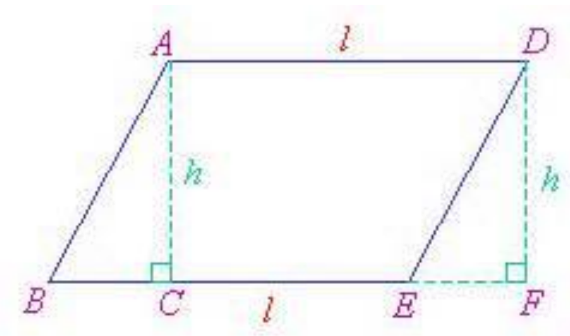


AREA OF A RECTANGLE: A = LW

The area of a square can also be found by applying the formula for the area of a rectangle. A square is a rectangle with four congruent sides. If each side of the square is represented by the variable S, using the formula A=lw and substituting S for both length and width, then $A = S \cdot S = S^2$ is obtained. Therefore the formula for the area of a square is $A = S^2$.

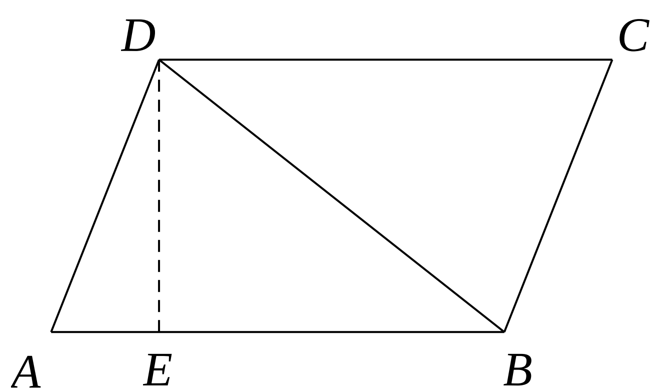
AREA OF A SQUARE: $A = S^2$

The formula for the area of a parallelogram can be derived from the area of a rectangle by taking the right triangle from one side of parallelogram and placing it onto the other side to form a rectangle. As shown in the diagram on the next page of the parallelogram ABED, the right triangle ABC can be removed and placed on the right side forming right triangle DEF. The length BE is the base of the parallelogram with AC being its height. After moving triangle ABC to the position of DEF a rectangle of equal area is formed. Segment AC is equal to DF which are the height and also the width of the rectangle ADFC. The base BE is also congruent to CF since BC and EF are congruent. *BE* = *BC* + *CE*, by the segment addition postulate and *BE* = *CE* + *EF* by substitution. *CF* = *CE* + *EF*, so by substitution *BE* = *CF*. The base of the parallelogram is equal to the width of the rectangle. Therefore, since the area of a rectangle is *A* = *LW* and the base of the rectangle is equal to the length of the rectangle and the height of the parallelogram is equal to the width of the rectangle the area of a parallelogram is equal to base times height. *A* = *bh*.



AREA OF A PARALLELOGRAM: A = bh

The formula for the area of a triangle can be derived by drawing a diagonal in a parallelogram. The opposite sides of a parallelogram are congruent. Using parallelogram ABCD, in the diagram to the right, by drawing diagonal BD we create two congruent triangles ABD and CDB. The tringles are congruent using the SSS postulate, since the opposite sides of a parallelogram are congruent and BD = BD by the reflexive property. Since the triangles are congruent the area of one triangle is equal to half of the area of the parallelogram. Therefore the area of a triangle is equal to half of the base times its height. *A* = ½*bh*.



AREA OF A TRIANGLE: A = ½bh

Below is an activity that the students can complete to prove the formulas for the areas of a parallelogram and triangle by using grid paper, a ruler, and scissors.

1. Draw two large parallelograms using the condition*: If two line segments are parallel and congruent then the quadrilateral formed is a parallelogram*. Make two of the sides horizontal.

2. Draw two altitudes in each parallelogram from a vertex to a horizontal side.

3. Label each horizontal side with a “b” and each height with an “h”. Place the labels in the interior of each parallelogram.

4. Cut out each parallelogram.

5. Using the first parallelogram make one cut along a height. By rearranging the pieces can you make another figure that you know the area of?

   Figure:

   Area:

   Since the area of the parallelogram and your new figure are the same, what can you conclude about the area of a parallelogram?

   Area =

6. Using the second parallelogram. Make one cut to form two triangles.

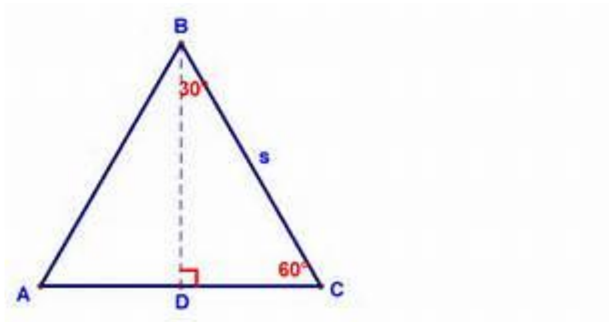
7. What is true about the two triangles?

8. Compare the area of each triangle to the area of the parallelogram.

9. What can you conclude about the area of a triangle? What is the formula to find the area of a triangle?

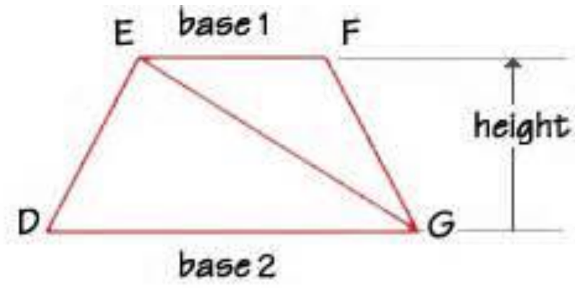
Area =

The next figure is that of a special triangle that has a formula of its own. This is an equilateral triangle. An equilateral triangle is also equiangular. So not only does it have all of its sides congruent, but so are all of its angles. Which means that the equilateral triangle is a regular polygon. Since, all the angles are congruent each angle is equal to 60^o. First, draw an altitude, from one of the vertices, point B in our diagram to side AC. This altitude also creates two congruent right triangles, by using the HL Theorem as $BC = AB$ and $BD = BD$. Angles ABD and CBD are congruent because they are corresponding parts in congruent triangles, also making them each 30^0. AC is bisected by the altitude as well because AD and DC are corresponding parts. So, $AD = \frac{1}{2}AC$. We can use the Pythagorean Theorem to find the height, BD, of the triangle. Let BC be s, then $BC = AC = s$. So, $AD = \frac{1}{2}s$. By applying the Pythagorean Theorem to triangle BDA, $BD^2 + (\frac{1}{2}s)^2 = s^2$. By applying algebraic properties $BD^2 = s^2 - \frac{1}{4}s^2 = \frac{3}{4}s^2$. Finally by raising both sides to the power of ½, we obtain the equation $BD = \frac{\sqrt{3}}{2}s$. Now, using the formula for the area of a triangle we obtain the formula $A = \frac{1}{2}\cdot\frac{\sqrt{3}}{2}s\cdot s$ which simplifies to the area formula of an equilateral triangle $A = \frac{\sqrt{3}}{4}s^2$.

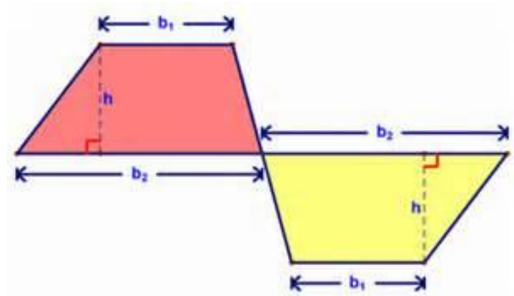


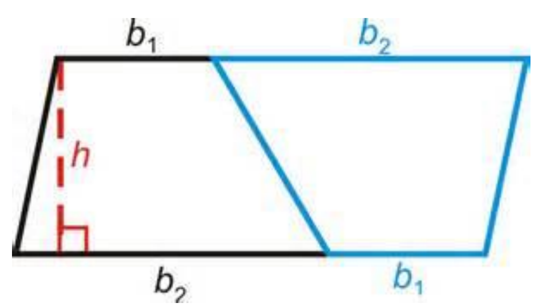
AREA OF AN EQUILATERAL TRIANGLE: $A = \frac{\sqrt{3}}{4}s^2$.

The formula for the area of a trapezoid can be derived from the area of a triangle. By drawing a diagonal, EG in the diagram, the area of the trapezoid is divided into two triangles. The triangles have the same height since EF and DG are parallel. Let EF = b_1 and DG = b_2 and the height = h. The area of $\Delta EFG = \frac{1}{2}b_1h$ and the area of $\Delta DEG = \frac{1}{2}b_2h$. The area of the trapezoid is the sum of the areas of the two triangles, thus $A = \frac{1}{2}b_1h + \frac{1}{2}b_2h$. Then by factoring out the greatest common factor of ½h we obtain a final formula of $A = \frac{1}{2}h(b_1 + b_2)$.



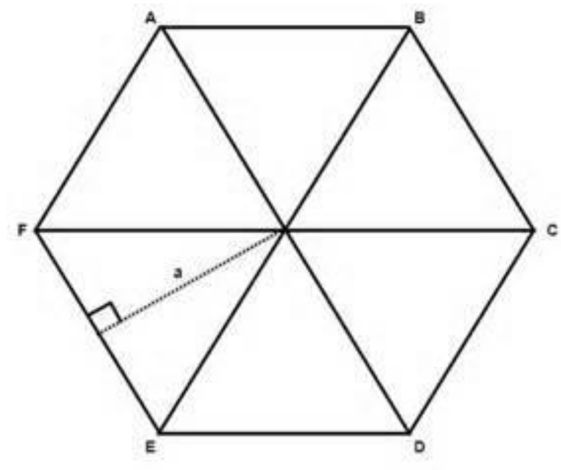
The formula for a trapezoid could also be derived by using parallelograms. A congruent trapezoid can be formed by rotating the original trapezoid 180° around the lower left vertex (see the first figure), then translating the new image up and to the right so that the two sides match forming a trapezoid (see the second figure). The area of the trapezoid is now formed by two congruent trapezoids, so the area of the parallelogram is equal to double the area of the original trapezoid. The area of the parallelogram is equal to $h(b_1 + b_2)$. This is twice the area of the trapezoid. Therefore, two times the area of the trapezoid equals $h(b_1 + b_2)$. So, by multiplying both sides by ½ we obtain $A = \frac{1}{2}h(b_1 + b_2)$.





AREA OF A TRAPEZOID: $A = \frac{1}{2}h(b_1 + b_2)$

The formula for the area of a regular polygon, with n sides, can be derived by using triangles. When a regular polygon is inscribed in a circle, the radius of the circle is the distance from the center of the polygon to one of its vertices. The polygon can then be divided into congruent isosceles triangles by drawing a radius to each of its vertices. This will create the same number of congruent isosceles triangles as there are sides in the polygon. To find the area of one of the triangles formed, an altitude also called the apothem of the polygon must be drawn which is also a median in the triangle. This can proved using the fact that the triangles are congruent. When the altitude is drawn it forms two right triangles, the triangles are congruent by the Hypotenuse-Leg Theorem. Then by corresponding parts the altitude bisects both the base and the vertex angle of the isosceles triangle making the altitude a median in the triangle as well. To find the area of the triangle we must first find the vertex angle. To do this, divide 360^o by the number of triangles formed. The apothem bisects the vertex angle. Let s = the length of the side of the polygon. Then the length of the side of the right triangle that is also part of the side of the polygon will be $^1/_2$s. By using the trigonometric function tangent, the length of the apothem can be determined, which is also the height of the triangle. To find the area of the triangle the formula, $^1/_2$bh can be used. The height is the apothem and the base is the length of the side of the polygon. So, $A= {}^1/_2as$. There are n triangles in the n-sided polygon, therefore multiplying the area of the triangle by the number of triangles n gives the area of the regular polygon is $A = {}^1/_2ans$. The perimeter of the regular polygon is the number of sides times length of a side, $p = ns$. By substituting this into the formula, the simplified formula for the area of a regular polygon is $A = {}^1/_2ap$.



AREA OF A REGULAR POLYGON: A = ½ap

3. PI AND THE CIRCLE

What is pi? Pi is a well-defined constant. It is well defined by similarity as all circles are similar. Pi is defined as follows:

Pi ***is circumference divided by diameter.***

Where did pi come from? The earliest known recording of pi was from about 1650 BC by Ahmes from Egypt. He claimed that if you take $^{8}/_{9}$ of the twice the radius and square it, the area of a circle with that diameter will be obtained. Using his statement, $A = (^{8}/_{9}2r)^2$, with some algebraic manipulations he obtained that pi = $^{256}/_{81} \approx 3.16049$. About 100 years later the Babylonians and Hebrews used 3 for pi. [3] Pi was also mentioned and talked about in the Bible in 2 Chronicles chapter 4 verse 2.

> *"He made the sea of the cast metal, circular in shape, measuring ten cubits from rim to rim and five cubits high. It took the line of thirty cubits to measure around it."* [10]

This states that the diameter is 10 cubits and the circumference is 30 cubits. Using the ratio that pi is equal to circumference divided by diameter, it can be calculated that three was used for pi.

Pi was left at that approximation for several years until about 430 BC. Anitphon and Bryson of Heraclea attempted to find the area of a circle by exhaustion. They inscribed polygons in a circle. They started with a hexagon and then doubled the number of sides to a dodecagon. They continued doubling the number of sides in the polygon but they made little progress. Then about 200 years later Archimedes used the some concept of exhaustion but focused on perimeters instead of area. Using both inscribed and circumscribed 96 sided polygons he was able to claim that pi was between $3^{1}/_{7}$ and $3^{10}/_{71}$. The average of these two values is about 3.1419 which is less than 0.003 units from the true value of pi. [3]

More than 650 years after Anitphon and Bryson, Lui Hui of China used the same method of exhaustion with area and inscribed polygons. He was able to approximate pi to be 3.1416 using a 3,072 sided polygon. Tsu Ch'ungchih and his son Tsu Keng-chih were able to use a 24,756 sided polygon. They achieved an approximation of $^{355}/_{113}$ or 3.1415929 for pi. No one found a more accurate value for pi for more than 1000 years. [3]

Francais Viete used the Archimedean Method of exhaustion using perimeters to approximate pi. In 1579, he used a 393,216 sided polygon to approximate pi accurately to 10 decimal places. His biggest step toward getting the value for pi came when he described pi as an infinite product. He broke the polygons into triangles and used the ratio of the perimeters between the regular polygon and the second polygon with twice the number of sides and the cosine of the angle using the half angle formula.

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \times \frac{\sqrt{2+\sqrt{2}}}{2} \times \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \times \cdots \quad [3]$$

Then in 1655, John Wallis used an infinite product that was much more efficient than Francais Viete's. He used the area of ¼ circles and small rectangles to come up with the product below

$$\frac{\pi}{2} = \frac{2}{1} \times \frac{2}{3} \times \frac{4}{3} \times \frac{4}{5} \times \frac{6}{5} \times \frac{6}{7} \times \frac{8}{7} \times \cdots \quad [3]$$

In 1671, James Gregory and Gottfried Wilhelm Leibenz were able to write a series to approximate pi using the arctan formula. The series that Gregory discovered was simplified by Liebeniz in 1674. Leibeniz's series to approximate for pi uses the angle $^{\pi}/_{4}$. His series is listed below:

$$\frac{\pi}{4} \approx 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots$$

Then, by multiplying both sides of the equation by four an approximation for pi is achieved.

$$\pi \approx 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \frac{4}{11} + \cdots$$

While using this formula in 1794, Georg Vego calculated pi to 140 digits. [3]

Now using computers pi has been calculated accurately to over 51.5 billion digits.

But, why is the 16^{th} letter of the Greek alphabet, π, used to represent the value of pi? In the early days, c or p were used to represent the ratio of pi. In 1706, William Jones used π to represent pi for the first time in a book that he published. He however was not very influential in the mathematical world. However, Leonhard Euler was very influential and he started using the symbol in 1737. With Euler using the symbol, the other mathematicians started using the symbol π as well for pi. By 1794 most all of the mathematicians in Europe were using π to represent the value of pi. [3]

Before deriving the formula for the area of a circle, the students will be asked what exactly is pi? What does it represent? How may the number pi have been originally developed? The students will complete an activity using string, a pencil, a piece of paper, scissors, and a ruler. The students can be taken to a parking lot to discover what pi represents.

1. Tie a piece of string to the pencil, place a point on the paper to represent the center of the circle and make a circle using the string as the radius.

2. Measure the length of the diameter.

3. Cut a piece of string to the exact length of the diameter.

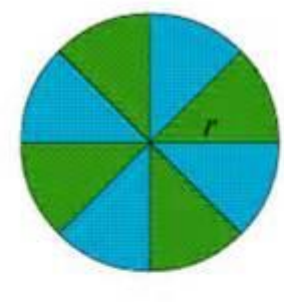
4. Take the string and measure the circumference of the circle using the length of the diameter as a unit.

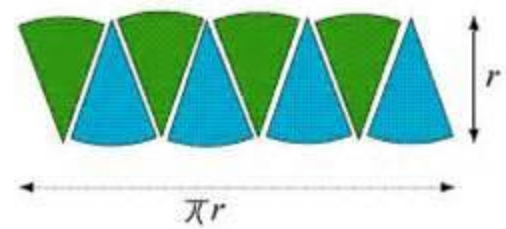
5. State the number of diameters needed to complete the length of the circumference of the circle.

6. Repeat this procedure several times with different sized circles.

7. What can you conclude about pi and the number of diameters needed to make the circle? Did the size of the circle effect the number of diameters?

Lastly, the formula for the area of a circle will be derived. To derive the formula for the area of a circle, divide a circle into congruent sectors. Then arrange the sectors as shown in the diagram below to approximately form a rectangle. As the number of sectors grows the object becomes more like a rectangle. This will introduce, in a very basic sense, the concept of a limit to the students. When the limit of the number of rectangles approaches infinity the shape of the sectors approaches a rectangle. The width of the rectangle will approach the radius and the length will approach half of the circumference. It is half of the circumference because half of the sectors are creating the length of the rectangle. To find the area of the rectangle we use the formula $A = lw$. So, the formula of $A = \frac{1}{2}Cr$ will be obtained. Circumference C is defined as $2\pi r$. By substituting this into the area formula, the formula $A = \frac{1}{2} \cdot 2\pi r \cdot r$ is achieved which simplifies to $A = \pi r^2$.



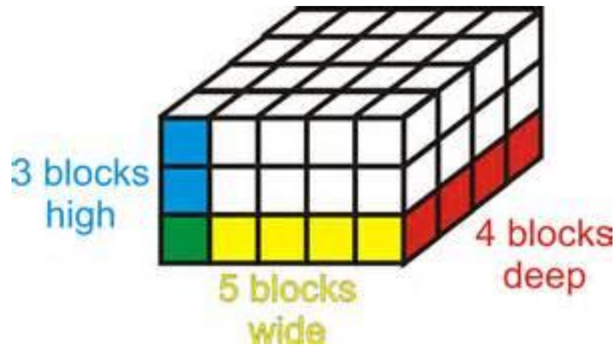


The formula for the area of a circle can also be obtained by using the limit process by using an inscribed polygon. By allowing the number of sides of the inscribed polygon to approach infinity it will become very close to a circle. By applying the formula for a regular polygon $A = \frac{1}{2}ap$, the perimeter will approach the circumference of the circle. The apothem will also approach the radius in the limit process. By substituting C for p and r for a the formula $A = \frac{1}{2}rC$ is obtained. As before, this gives $A = \pi r^2$.

AREA OF A CIRCLE: $A = \pi r^2$

4. THE BASIC VOLUME FORMULAS

The volume of a rectangular prism is probably the easiest to visualize. A rectangular prism can be formed by stacking cubical unit blocks as shown in the figure. By counting the unit blocks in the figure, the volume of the rectangular prism will be found. The number of unit blocks covering the base of the prism can be calculated by multiplying the length of the base by its width since the base is a rectangle. The unit blocks on the base are then stacked three times in our figure to form the prism. So the volume, the number of unit blocks, can be found by multiplying the area of the base by the height of the prism. Seeing that all prisms are built by stacking layers of bases, the volume of any prism can be found by using the formula $V = Bh$, where B is the area of the base. Since the base is a rectangle its area can be calculated using the formula for the area of a rectangle, $A = lw$, therefore the volume for a rectangular prism is $V = lwh$.



VOLUME OF A RECTANGULAR PRISM: V = LWH

The students can complete the following activity to develop the formula for the volume of a rectangular prism by using unit cubes as described above.

1. Build the following rectangular prisms and count the number of blocks needed to complete each prism.

2. Complete the table below:

L	W	H	Number of Blocks
3	2	2	__________________
4	3	1	__________________
5	2	2	__________________

3. Using the length, width and height of each prism, how can you calculate the number of blocks necessary to build the prism?

4. Build another prism and check to see if your conjecture works.

5. Explain why you believe that this may work for any rectangular prism.

The next figure that a formula for volume can be derived is a special rectangular prism, the cube. The formula for the volume of a cube can be derived from the volume of the rectangular prism. The cube is a rectangular prism with all edges being congruent. Let each edge be s, then length, width, and height would all be equal to s. By substituting s into the formula for the volume of a rectangular prism $V = lwh$, the formula $V = s \cdot s \cdot s$ is achieved, which simplifies to $V = s^3$.

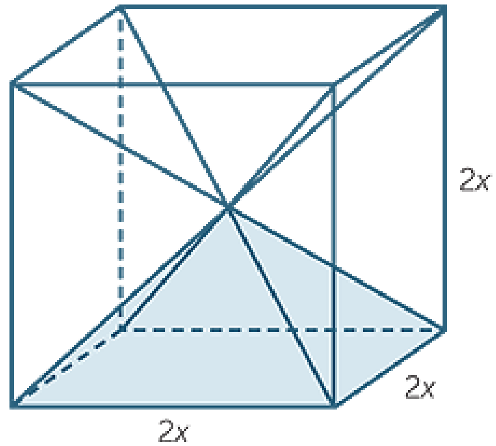
VOLUME OF CUBE: $V = s^3$

The formula for the volume of a cylinder can be derived quickly because a cylinder can be thought of as a prism with a circular base. A cylinder can be made of stacked congruent circles as a prism is made of stacked congruent polygons. The volume of a prism is V = Bh, since the base is a circle then the base area B is the area of a circle, $A = \pi r^2$. By substitution the volume of a cylinder can be found by using the formula $V = \pi r^2 h$.

VOLUME OF CYLINDER: $V = \pi r^2 h$

The volume of a cone can be found by using calculus and rotation around the x-axis to form a solid cone. Geometry students though have yet to take calculus and would not understand the integration necessary to prove the volume of a cone using this process.

So, another process is needed to derive the formula not using calculus. A cone can be thought of as a pyramid with a circular base, so if the formula for the volume of a pyramid can be derived then the formula for the cone can be derived. The volume of a pyramid can be derived using calculus and cross sections which geometry students will not understand. So, another method to derive the formula without calculus is needed.



To derive the volume of a pyramid, without calculus, notice that six pyramids can be inscribed in a cube with each face of the cube being the base of a pyramid and the vertex of each pyramid being at the center of the cube. The height of each pyramid will be half the length of a side of the cube. All six pyramids will then be congruent to each other since their bases and heights are the same. Let each side of the cube be 2x, as shown in the diagram, the height of each pyramid will be x and the base of each pyramid will be $2x \cdot 2x$. Thus, the area of the base of each pyramid will be $4x^2$. The volume of the cube is then $V = (2x)^3 = 8x^3$. Since all the pyramids are congruent and are inscribed in the cube the volume of each pyramid is then $^1/_6$ the volume of the cube. So, the volume of each pyramid is $^1/_6(8x^3)$. The formula can be rewritten as follows $V = {}^1/_6 \cdot 2 \cdot 4x^2 \cdot x$. Since the height of the pyramid h is equal to x and the base area B of the pyramid is $4x^2$, by substitution the formula $V = {}^1/_6 \cdot 2 \cdot B \cdot h$ is obtained. After simplifying the formula for the volume of a pyramid is $V = {}^1/_3 Bh$.

VOLUME OF A PYRAMID: $V = {}^1/_3Bh$

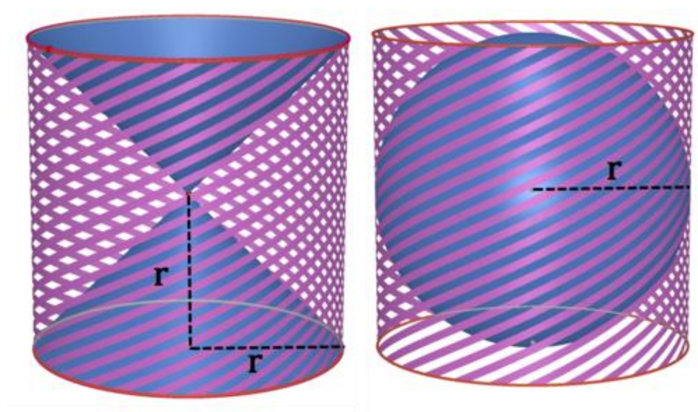
The volume formula for a cone can now be derived using the formula for a pyramid. The cone has a circular base which has an $A = \pi r^2$. So, by substituting this for base area in the volume formula for a pyramid the formula for the volume of a cone is $V = {}^1/_3\pi r^2 h$.

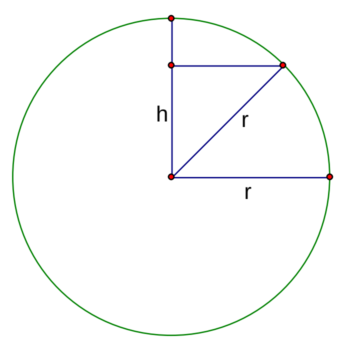
VOLUME OF A CONE: $V = {}^1/_3\pi r^2h$

5. THE SPHERE

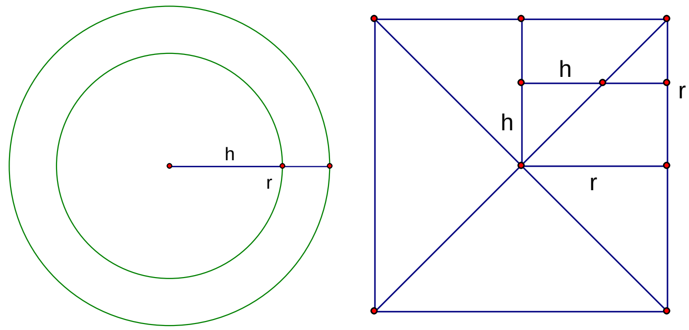
The final formula for volume that will be derived is for the sphere. Again the volume formula for a sphere can be derived using calculus and rotating a semicircle around the x-axis. In pre-calculus and geometry courses the formula needs to be derived without integration Cavalieri's Principle be used to derive the formula. The principle states that if the areas of the cross sections of two solids are equal and the height of the two solids are equal then the volumes of the two solids are equal. [4]

To derive the formula for the volume of a sphere two objects will be given. The first is a sphere with a radius of r inscribed in a cylinder and two inverted cones each with their radii and heights being equal to r inscribed in a cylinder. First, take a cross sectional slice across the sphere and the cones inscribed in the cylinders that is h units from the center. The cross section of the sphere will be a circle, as shown in the figure below. The radius of the cross section of the sphere is calculated using the Pythagorean Theorem. One leg of the right triangle is h, the other is the radius of the cross section, and the hypotenuse is r. Using the Pythagorean Theorem the radius of the cross section will be $\sqrt{r^2 - h^2}$. So, the area of the cross sectional circle is $A = \pi(\sqrt{r^2 - h^2})^2$ which simplifies to $A = \pi(r^2 - h^2)$. The cross section of the cone inscribed in the cylinder is a washer, as shown in the figure on the next page. By looking at the front view of the two cones in the cylinder it will be noticed that the cones form right isosceles triangles, the radius and the height both have length r.



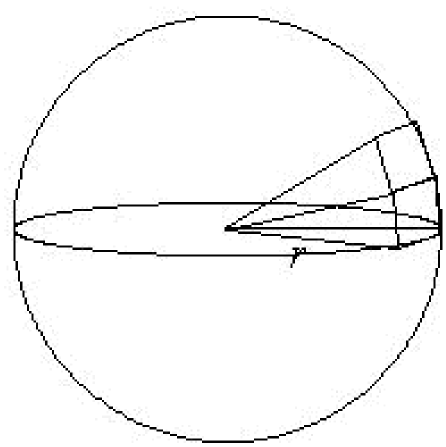


By taking the cross section, h units from the center of the cylinder, the radius of the cross sections of the cone will be h and the radius of the cylinder will be r. The vertical cross section through the cylinder and cones is a square. The horizontal cross sections have the shape of a washer or annulus with the cones corresponding to the inner circle and the cylinder corresponding to the outer circle. So, the area of the horizontal cross section will be $A = \pi r^2 - \pi h^2$. By factoring out the greatest common factor of pi the formula $A = \pi(r^2 - h^2)$ will be achieved for area. This is the same area as the cross section of the sphere also taken h units from the center. So, when all the cross sections of each figure are combined, the volume of the sphere will be equal to the volume of the cylinder minus the volume of the two cones. Using the volume formulas for the cylinder and cone derived earlier, a formula for the volume of a sphere will be obtained as follows, $V = \pi r^2 h - 2(^1/_3 \pi r^2 h)$ where h is the height of the cones. The height of the cylinder is 2r and the height of the cone is r. By substituting these values for h into the formula we obtain the formula $V = \pi r^2(2r) - 2(^1/_3 \pi r^2(r))$ which then simplifies to $V = 2\pi r^3 - {}^2/_3 \pi r^3$. Thus, $V = {}^4/_3 \pi r^3$ is the volume formula for a sphere.



VOLUME OF SPHERE: $V = {}^4/_3 \pi r^3$

Next, the formula for the surface area of a sphere will be derived. This can be completed now that the formula for the volumes of a sphere and pyramid have been discovered. The sphere will be filled with pyramids in order to approximate its surface area. The vertex of each pyramid will be at the center of the sphere. The volume of the pyramid is equal to $^1/_3$Bh, where B is the area of the base of the pyramid. The volume of the sphere, can be realized by approximating the interior of the sphere by packing an infinite number of pyramids. The volume of the sphere can be realized as the limit as the number of pyramids approaches infinity of the summation of the volumes of the pyramids. The height of each pyramid will approach the radius of the sphere. Using this process the volume of the sphere will be $^1/_3$Ar, where A is the limit of the sum of the bases of all the pyramids which is the surface area of the sphere. This formula is equal to the formula that was



derived earlier for the volume of a sphere giving the equation ${}^{4}/_{3}\pi r^{3} = {}^{1}/_{3}Ar$. By multiplying both sides of the equation by 3 and dividing both sides by r the formula for the surface area of the sphere, $A = 4\pi r^2$ is derived.

SURFACE AREA OF SPHERE: $4\pi r^2$

6. THE PYTHAGOREAN THEOREM

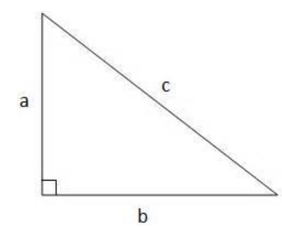
There is another formula that is very important in geometry. It is used in solving many problems that involve area and volume. This formula is the Pythagorean Theorem. It is stated as follows: [8]

The Pythagorean Theorem states that in a right triangle the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.

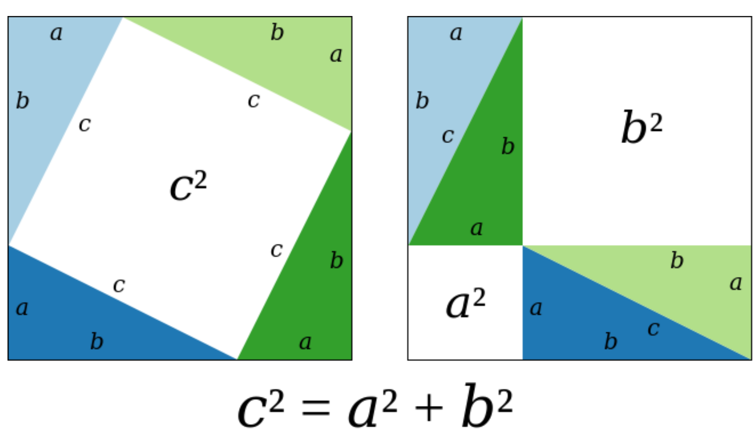
$Hypotenuse^2 = Leg_1^2 + Leg_2^2$

Given the right triangle below, the Pythagorean Theorem is also quite often stated following the definition using the triangle below obtaining the formula below:

$c^2 = a^2 + b^2$



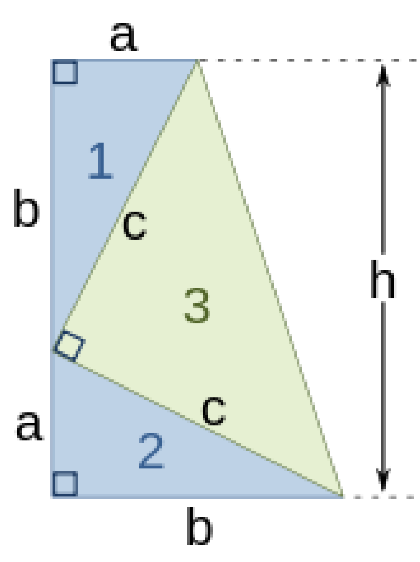
The Pythagorean relationship was known by the Babylonians about 1600BC. [1] At this time the relationship had yet to be proved. Pythagoras, who lived from about 569 BC to 500 BC, imported proof into mathematics. This was probably his greatest achievement. [2] He provided a proof of the Pythagorean Theorem. His proof consisted of the rearrangement of triangles as shown in the diagram to the right. The two large squares are identical and the four triangles are all



congruent. The only difference is the white regions, thus giving the result of $c^2 = a^2 + b^2$. [9]

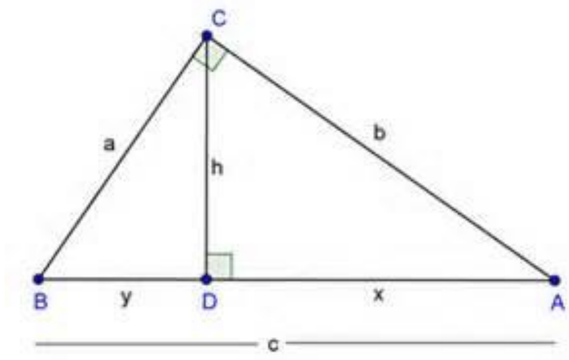
This is just one proof of this famous theorem. Another way to prove The Pythagorean Theorem involves algebra. It uses the left side of the diagram on the previous page. The length and width of the large square are a + b. So, the area of the large square is $(a + b)^2 = a^2 + ab + b^2$. The area of the square can also be found by adding the area of the four triangles to the area of the small square. The four triangles are all congruent by the SAS postulate. The area of each triangle is ½ab, so the area of all four triangles combined is $4(½ab) = 2ab$. The area of the small white square is c^2, since each side is c. So the area of the square obtained by adding all the triangles and the small square is $2ab + c^2$. By taking the two calculations for the area of the square and setting them equal to each other the equation $a^2 + 2ab + b^2 = 2ab + c^2$ is obtained. Subtract 2ab from both sides and the Pythagorean Theorem $a^2 + b^2 = c^2$ is achieved with c representing the length of the hypotenuse and with a and b representing the lengths of the legs of the right triangles.

The Pythagorean Theorem was also proved by United States President James A. Garfield, then a United States Representative. He proved the Pythagorean Theorem using a trapezoid instead of squares. [9] He found the area of the entire trapezoid using the formula for the trapezoid. The height is a + b and one base is a while the other is b. By using the formula for the area of a trapezoid the equation $A = ½(a + b)(a + b)$ is obtained. This is equivalent to $A = ½(a^2 + 2ab + b^2)$. Next, the area of the trapezoid can be found combining the area of its parts. Triangles 1 and 2 are congruent by the SAS postulate. The area of each of those triangles is ½ab while the area of the third triangle is $½c^2$. By adding the results of all three triangles the formula $A = ab + ½c^2$ is obtained. Then take the two results and set them equal to each other obtaining the equation $½(a^2 + 2ab + b^2) = ab + ½c^2$. Multiply both sides by 2, the



result is $a^2 + 2ab + b^2 = 2ab + c^2$. Then subtract 2ab from both sides and the result is the conclusion of the Pythagorean Theorem $a^2 + b^2 = c^2$.

A forth way to prove the Pythagorean Theorem is by using similar triangles. In the diagram below there are three similar triangles, ΔABC ~ ΔACD ~ ΔCBD all by the AA postulate. Two of the similar triangles will be used at a time. First, using ΔABC ~ ΔACD, two of the sets of proportional sides give the equation $^{c}/_{b} = {}^{b}/_{x}$. By multiplying both sides by bx, the equation $cx = b^2$ is achieved. Second, using ΔABC ~ ΔCBD, two of the sets of proportional sides give the equation $^{c}/_{a} = {}^{a}/_{y}$. By multiplying both sides by ay, the resulting equation is $cy = a^2$. Next, add the two equations together obtaining $cx + cy = a^2 + b^2$. This is equivalent to the equation $c(x + y) = a^2 + b^2$. But, $x + y = c$ so by substitution $c \cdot c = a^2 + b^2$ is the result and therefore $c^2 = a^2 + b^2$.



There are many other ways to prove the Pythagorean Theorem. The previous four are proofs that geometry students would be able to understand fully. By proving this theorem the students will be able to understand where it comes from and why it works when they use it in solving many different types of problems including area and volume problems and applications.

7. CONCLUSION

Understanding the history and the origins of the concepts of area and volume and their formulas is imperative to the comprehension and appreciation of the meaning of the solutions the students achieve to applied problems. Demonstrating the proofs for the formulas of each figure combined with the historical perspective will enhance student learning. Students will be better able to recall and understand their calculations and what they represent allowing them to be more proficient at problem solving. They will be able understand the meaning of their solutions and will gain the ability to improve their mastery of area and volume.

With an improved understanding of area and volume, the students will be more successful and confident in their mathematical studies leading to more excitement about mathematics. The students will be able to use these concepts later in their mathematical studies to enhance their success in future mathematical courses. Students will be able to visualize how these concepts are used thereby gaining the fundamental knowledge base for selected coursework in later courses such as calculus. This confidence and understanding will allow the students to have more success in college or career endeavors.

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