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Infinity in the High School Mathematics Classroom

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Infinity in the High School
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1. INTRODUCTION

In general, infinity represents a large quantity that is greater than any imaginable real number. When describing infinity to young children or even to students, it is often stated that if you name a number, I can name a larger one. This process will continue forever and as a result we arrive at the concept of infinity. There is a more formal, mathematical definition of infinity. "Infinity is represented by the symbol ∞, first used by John Wallis in his book *De sectionibus conicis* (Of conical sections), published in 1655" [13]. The formal definition of infinity is given as follows [12]:

$$\bullet \, \forall n \in \mathbb{Z} : n < \infty$$
$$\bullet \, \forall n \in \mathbb{Z} : n + \infty = \infty$$
$$\bullet \, \forall n \in \mathbb{Z} : n \times \infty = \infty$$
$$\bullet \, \infty^2 = \infty$$

The first statement translates to: for all numbers in the set of integers, all of the numbers are less than infinity. This means that infinity is greater than any number in the set of integers. The set of integers is said to be infinite meaning that there is no end to the numbers that can be written for the set. The second statement says: for all numbers in the set of integers, any number plus infinity is equal to infinity. This means that infinity is so large that adding any number in the set of integers will have no effect on the value. The third statement says: for all numbers in the set of integers, any number times infinity is equal to infinity. Similar to the result for addition, infinity is so large that multiplying it by any number in the set of integers will have no effect on the value. Finally, the fourth statement says that infinity times infinity is equal to infinity. Again, infinity is so large that multiplying it by itself has no effect on the value.

The idea of infinity arose at the time of the Ancient Greeks who did not define infinity formally but as a philosophical concept [9]. In 250 BCE Archimedes method of calculating the area of a circle used the idea of limits and infinity. Archimedes inscribed and circumscribed polygons about a circle. He started by finding the area of a six-sided

figure. He then doubled the number of sides to find the new area and continued this process until finally calculating the area of a 96-sided figure. "Here, Archimedes encountered two concepts which would become hugely popular later – that of limits and that of infinity, for the perfect area would be given by a polygon with infinitely many sides, so the two polygons would converge at that point. As the number of sides tends towards infinity, the difference between the area of the polygons and the area of the circle tends towards zero and the limits coincide" [13]. Later in the 1600's mathematician Gottfried Leibniz and scientist and mathematician Sir Isaac Newton independently discovered calculus. Leibniz was interested in the properties of summing infinitesimals. "His work treated continuous quantities as though they were discrete, a logical flaw that he and others overlooked – even though it was an issue so old that it was the difficulty at the heart of Zeno's paradoxes" [13]. However, Newton's calculus allowed for the expression of infinite sums through power series which are infinite.

In 1873, German mathematician Georg Cantor showed that the set of rational numbers are countable despite the fact that the set itself is infinite [19]. Cantor was able to show that the size of the set of natural numbers was the same size as the set of integers through one-to-one correspondence. The correspondence goes as follows:

$$\begin{array}{ccccccc} ... \ 7 & 5 & 3 & 1 & 2 & 4 & 6 \ ... \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ ...-3 & -2 & -1 & 0 & 1 & 2 & 3... \end{array}$$

Cantor created a relationship in which 1 was match to 0, each of the even numbers in the set of natural numbers was matched to the positive numbers in the set of integers and the odd numbers in the set of natural numbers was matched to the negative numbers in the set of integers. Cantor also showed that the set of natural numbers was the same size as the set of rational numbers. This one-to-one correspondence can be shown in the image on the next page.

	1	2	3	4	5	6	7	8	...
1	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	...
2	$\frac{2}{1}$	$\frac{2}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$	$\frac{2}{6}$	$\frac{2}{7}$	$\frac{2}{8}$	...
3	$\frac{3}{1}$	$\frac{3}{2}$	$\frac{3}{3}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{6}$	$\frac{3}{7}$	$\frac{3}{8}$	...
4	$\frac{4}{1}$	$\frac{4}{2}$	$\frac{4}{3}$	$\frac{4}{4}$	$\frac{4}{5}$	$\frac{4}{6}$	$\frac{4}{7}$	$\frac{4}{8}$	...
5	$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{5}{5}$	$\frac{5}{6}$	$\frac{5}{7}$	$\frac{5}{8}$	...
6	$\frac{6}{1}$	$\frac{6}{2}$	$\frac{6}{3}$	$\frac{6}{4}$	$\frac{6}{5}$	$\frac{6}{6}$	$\frac{6}{7}$	$\frac{6}{8}$	...
7	$\frac{7}{1}$	$\frac{7}{2}$	$\frac{7}{3}$	$\frac{7}{4}$	$\frac{7}{5}$	$\frac{7}{6}$	$\frac{7}{7}$	$\frac{7}{8}$	...
8	$\frac{8}{1}$	$\frac{8}{2}$	$\frac{8}{3}$	$\frac{8}{4}$	$\frac{8}{5}$	$\frac{8}{6}$	$\frac{8}{7}$	$\frac{8}{8}$	...
$\vdots$	$\vdots$								

Finally, Cantor was able to show that the set of real numbers, the set that includes both rational and irrational numbers, is not the same size as the set of natural numbers. This is due to the fact that no one-to-one correspondence can be shown between the set of real numbers and the set of natural numbers. Therefore, the set of real numbers cannot be equal in size to the set of integers or the set of rational numbers. This means that there are different sizes of infinity.

Despite the fact that infinity is a complex idea, it is a staple in mathematics and is seen in almost every concept. The concept of infinity is one that forces people to stop and think. As a child, you play the game of "I love you this many." The game starts when one person spreads their arms to demonstrate some measurable amount of love. The other person counters with "No, I love you more." This bantering goes on until someone says, "No, I love you infinity." This brings up a new sense of measurement that one cannot see, but surely it is quite a large amount of love. The final few exchanges go as follows, "No, I love you infinity plus one. Well I love you infinity times infinity." At a young age one believes that multiplication will give you the largest result as you have not yet learned about exponential functions. Where does this idea of infinity come from? As a child, how do we have a sense of what infinity means? At this age, infinity just means some large (finite) number that we do not know how to count to yet. Sure, it has

been explained that infinity means you can count forever; but surely there has to be a stopping point.

After finishing my seventh year of teaching, I realize that students are exposed to problems and mathematical ideas that often involve infinity. In algebra I, we graph lines and quadratics on the coordinate plane and discuss the importance of drawing arrows at the ends of the graphs of the functions to indicate that the graph of the function will go on forever. In precalculus we discuss infinite series and introduce the concept of limits. We discuss the end behavior of graphs of polynomial and rational functions and how the end behavior will tend towards infinity or negative infinity unless the domain is restricted. Yet, we never once formally define infinity. We remind them that infinity is not a number but represents a large quantity that is greater than any real number they can imagine. This does not mean they actually understand what it means when I say that an infinite series can have a finite sum. How is that possible?

The goal of my paper is to create lessons or discussions that will introduce students to the concept of infinity and see how it plays into the mathematics that they do. Since this concept is constantly found in the mathematics that I teach, I decided to create ways students can explore the idea. All of these ideas, lesson and discussions stemmed from different topics I was exposed to in the Masters of Arts Degree for high school teachers.

The first section will discuss the question of whether or not $0.999\ldots = 1$. Some students struggle with the truth of this equation because they do not fully accept that there are infinitely many nines. I found some discussion points that will be helpful in their acceptance of this true mathematical statement. The second section will introduce one of Zeno's Paradoxes in which Achilles and a tortoise are in a race. The question at the end of the story poses an interesting platform to discuss infinite series. The third section will look at the number e and how it is defined. Often in high school, students are introduced to the number e and told that it is a "natural number". However, they are not typically told how the number is mathematically defined. Two of the definitions of the number use the concept of infinity and working with the definitions will be helpful when using the number in application problems such as compound interest. The fourth section looks at a famous fractal called Koch's Curve. Students are asked to draw the curve and complete a

few iterations. They are then asked to find the perimeter and area under the curve after an infinite number of iterations. The result is surprising! The final section introduces the paradox called Gabriel's horn. In this problem, students rotate the graph of the function $f(x) = \frac{1}{x}$ about the x-axis and are asked to find the volume and surface area of this object. This problem has results similar to Koch's curve.

These different lessons and discussion will be implemented throughout the school year. These ideas will force students to think critically and outside of the box. Students will be given the opportunity to have rich mathematical discussions and their understanding of infinity will evolve.

2. PROOF THAT 0.999… = 1

The first time the question "Does 0.999… = 1" was asked of me, I was in a college math class and an undergraduate. I thought it was a trick question. Surely the answer is no. How is it possible that these two numbers are equal when one has infinitely many 9's and the other value has exactly one number: 1? The professor worked through a simple proof and I remember still having this feeling of being tricked. Perhaps the math the professor had done was not "math legal." Was the professor trying to test us to see if we could catch his mistake? In fact, the professor was correct in his statement and it took several different proofs for me to understand why. Because of my hesitancy in accepting the truth, I feel compelled to ask my high school students the same question and have them explore different proofs depending on their level of mathematical understanding.

After first posing the question to my students, both freshman level and senior level math students, I will give them a few minutes to let the question settle in their minds. After allowing time for peer discussion, we will discuss as a whole class whether or not 0.999… really does equal 1. The heart of this discussion will first center on what the … means after the 0.999. It is important for students to understand that the ellipsis after the nines means that the 9's in this decimal notation will continue repeating with no end. The number of digits is infinite. This means that the decimal will not terminate. Since infinity it not tangible, it is often difficult for students to understand the concept. Sure, the number will go on forever, but some students have the misconception that the number will eventually stop.

One way to direct this conversation is to give the number 0.9. Then ask the students if they can name a number between 0.9 and 1. After some thought, students will be able to give example such as 0.91 or 0.975 or 0.99. This will then lead to the next question: can the students name a number between 0.99 and 1? They should be able to give an example of 0.995 or 0.999. This questioning will continue on until the idea of

having an infinite number of nines arises. Once a student mentions the number 0.999…, the next question will be whether or not they can name a number between 0.999… and 1. This is an impossible task since, as stated above, 0.999… = 1. This argument may not be convincing enough for most students. Therefore, I will have a series of proofs laid out for them to work through.

The first proof I will use is the one my professor showed us in class. This method uses the idea of multiplying by ten. "When a number in decimal notation is multiplied by 10, the digits do not change but each digit moves one place to the left" [22]. First you define x such that $x=0.999...$, then multiply both sides of the equation by 10. Next, subtract the second equation from the first equation and divide the result by 9. The solution is that $x = 1$. This implies that $1=0.999...$ since both values are equal to x. The proof is shown below [16].

$$x=0.999...$$
$$10x=9.999...$$

Subtraction the second equation from the first equation:

$$10x=9.999...$$
$$\underline{-\ x=0.999...}$$
$$9x=9.000...$$
$$\downarrow$$
$$x=1$$

Some students might think that since we multiplied both sides by 10, and the "last" 9 in the number 0.999… was shifted, then the new "last" number would be 0. However, this is not the case since there is not a finite number of 9's. "There is no 'end' after which to put that zero" [16]. The operations in this proof are simple enough that it could be shown and discussed in an Algebra I class. A variation of this proof is shown below [22].

$$x = 0.999\ldots$$
$$10x = 9.999\ldots$$

We know that 9.999… is 9 greater than 0.999. Therefore, the proof continues as follows:

$$10x = 9 + 0.999\ldots$$
$$10x = 9 + x$$
$$9x = 9$$
$$x = 1$$

To move from the first step to the second step, we replace 0.999… with x since we originally defined the two to be equal. Next, subtract x from both sides and then divide both sides by 9. This is slightly different than the first proof in that the expansion of 0.999… is replaced by x before subtracting a term on both sides. The idea of the proof is exactly the same.

A few other proofs that students in Algebra I will be able to understand have to do with fractions. A simple way to show that 0.999… = 1 is to look at the pattern created when writing fractions with a denominator of 9 [22]. I would ask the students to write what $\frac{1}{9}$ equals. They will find that $\frac{1}{9} = 0.111\ldots$. I will then ask them to find the decimal form of fractions with denominators of 9 and numerators of 2 − 8. The results are as follows:

$$\frac{2}{9} = 0.222\ldots, \quad \frac{3}{9} = \frac{1}{3} = 0.333\ldots, \quad \frac{4}{9} = 0.444\ldots, \quad \frac{5}{9} = 0.555\ldots,$$
$$\frac{6}{9} = \frac{2}{3} = 0.666, \quad \frac{7}{9} = 0.777\ldots, \quad \frac{8}{9} = 0.888\ldots$$

These results are usually very easy for students to accept. Perhaps it is because these are fractions that do not equal an integer and so it is acceptable for them to be equal to some decimal expansion. The next natural question to ask the students would be,

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"based on the pattern, what decimal expansion can be written to equal $\frac{9}{9}$?" Students will probably be hesitant to give the answer of 0.999... because they have been trained to recognize that a number divided by the exact same number is equal to 1. However, this argument may help some students see that 0.999... = 1. This is a demonstration, not a formal proof.

If this demonstration does not convince the students, perhaps the next one will. It is connected to the previous argument. Earlier the students found that $\frac{1}{9}=0.111...$. Now I will ask them to multiply both sides by 9 to discuss the result. The answer is shown below [22].

$$\frac{1}{9}=0.111...$$
$$9\times\frac{1}{9}=9\times0.111...$$
$$1=0.999...$$

The assumption with this proof is that $9\times0.111...$ is equal to $0.999...$. When students are taught multiplication, they are taught to start the operation on the right side of the number. However, the number 0.111... is infinite and therefore cannot be multiplied starting on the right side. To show that $9\times0.111...=0.999...$, students could first check that $2\times\frac{1}{9}=2\times0.111...=0.222...$. Students should feel comfortable with this calculation since they had previously divided 2 by 9 to get $\frac{2}{9}=0.222...$. Students can continue this process by checking that $3\times\frac{1}{9}=3\times0.111...=0.333...$, $4\times\frac{1}{9}=4\times0.111...=0.444...$, and so on. Recognizing the pattern, it can be confirmed that $9\times\frac{1}{9}=9\times0.111...=0.999...$. This proof may be the easiest for the students to understand as it involves two very simple operations and there is little to argue about.

Another proof that uses division as the starting point begins with long division to show that $\frac{1}{3}=0.333....$ This is a fun exercise to remind students of the process of long division and to prove that $\frac{1}{3}=0.333...$ is similar to the calculator's truncated answer. The calculation is shown below [16].

$$
\begin{array}{r}
0.333 \\
3\overline{)1.0} \\
-9 \\
\hline
10 \\
-9 \\
\hline
10 \\
-9 \\
\hline
1
\end{array}
$$

This process continues on infinitely. Since the result in the quotient will always be 3, and the difference will always be 1, the process will continuously repeat. Recall that $\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=3\left(\frac{1}{3}\right)=1$. Due to the process of division above, then it should be reasonable to state that $0.333...+0.333...+0.333...=3(0.333...)$ should be equal to 1. However, $3(0.333...)=0.999....$ Therefore, $0.999... = 1$ [16].

The final proof that I will show my students uses geometric series. This is not a concept that Algebra I students will be able to understand, however it is a topic that is covered in Precalculus. The number $0.999...$ can be expanded and written as a series. This expansion is shown below.

$$
\begin{aligned}
0.999... &= .9+.09+.009+\cdots \\
&= \frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\cdots
\end{aligned}
$$

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This is a geometric series with a common ration of $\frac{1}{10}$. Using the sum formula for infinite geometric series, which is obtained using essentially the same method as the first proof, the sum of the terms is calculated as follows: $S = \frac{\frac{9}{10}}{1 - \frac{1}{10}} = \frac{\frac{9}{10}}{\frac{9}{10}} = 1$. Therefore, $0.999\ldots = 1$. This will be a great question to ask precalculus students after we have spent some time exploring the sums of infinite geometric series and it will be something fun for them to explore.

3. ZENO'S PARADOX

Zeno of Elea was a Greek Philosopher who wrote a book of paradoxes over 2,500 years ago [11]. Unfortunately, it has been stated that the "book has not survived, and what we know of his arguments is second-hand, principally through Aristotle and his commentators" [9]. Zeno believed that motion was an illusion and created four paradoxes on the matter [23]. One of his famous paradoxes is called "Achilles and the Tortoise." According to the story, Achilles, hero of the Trojan War, is in a race with a tortoise. He gave the tortoise a head start and the assumption is that Achilles is of course faster than the tortoise. In order for Achilles to win, he must first catch up to the tortoise. However, "before Achilles can catch the tortoise he must reach the point where the tortoise started. But in the time he takes to do this the tortoise crawls a little further forward. So next, Achilles must reach this new point. But in the time it takes Achilles to achieve this, the tortoise crawls forward a tiny bit further, and so on to infinity" [8]. Will Achilles ever overtake the tortoise [11]?

At first thought, the solution is obvious; of course Achilles will overtake the tortoise and win. However, if you read the paradox closely, and perhaps make a sketch of the situation, you will find that Zeno's argument could be logical. A possible sketch for the situation is shown below.
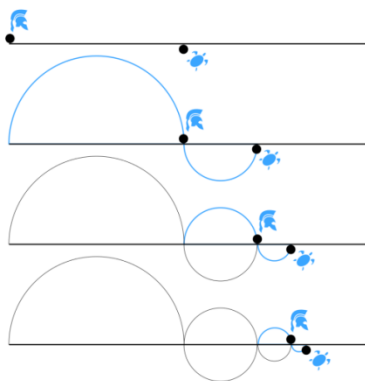
Notice that as Achilles is busy closing the gap from one point to the next, the tortoise has already moved on. In Zeno's argument, despite the fact that the tortoise is moving at smaller and smaller increments, and despite the fact that Achilles is moving at a

faster rate, Achilles will never catch up to the tortoise. Joseph Mazur, a professor emeritus of mathematics at Marlboro College states that "Achilles' task seems impossible because 'he would have to do an infinite number of 'things' in a finite amount of time'" [11]. This brings forth the notion of infinite series. There are two types of infinite series: a converging series and a diverging series.

An infinite series that converges is one that, when all of the terms are added together, the sum will equal a finite number. An excellent way to demonstrate a converging infinite series to students is by starting at one end of the classroom and walking to the other side. In order to get to the other side, you must first walk half the distance to the wall. Then, to continue the process, you walk half of the remaining distance. This process continues infinitely many times; will you ever make it to the other side of the classroom? This process can be represented by the infinite series shown below.

$$\text{Distance} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots + \frac{1}{2^n} + \cdots$$

As demonstrated in the classroom, you will have to take infinitely many small steps, but eventually, you would reach the wall given the assumption that as the distance between each step gets smaller, the time it takes to make each step is faster. If each step took the same amount of time to complete, you would never reach the wall because it would take infinite time. This means the time it takes to complete each step is at a faster rate, so the time is finite just like the distance. This means that by demonstration, the sum of the terms would equal one length of the room. Mathematically speaking, this series is a geometric series with a common ratio of $\frac{1}{2}$. Since the common ratio is greater than 0 and less than 1, the series converges using the formula $S = \frac{a_1}{1-r}$ where a_1 represents the first term in the series and r represents the common ratio. The solution to this series is

$$S = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1.$$

If Achilles "is making the gaps smaller at a sufficiently fast rate, so that their distances look more or less like this equation, he will complete the series in a measurable amount of time and catch the tortoise" [11]. In other words, if this series were matched up with the picture on the previous page, then eventually Achilles will catch the tortoise. This was shown mathematically using the sum formula for converging infinite geometric series. The fact that there are infinitely many terms in the series matches the fact that Achilles will have to close the gap infinitely many times. However, this infinite series has a finite sum which means that eventually Achilles will catch the tortoise and this opposed Zeno's argument.

Another scenario that opposes Zeno's argument is given as the following: "suppose that Achilles is running at 1 meter per second (*m/s*), that the tortoise is crawling at 0.1 *m/s* and that the tortoise starts out 0.9 *m* ahead of Achilles" [8]. Given this set up, one might believe that after 1 second, Achilles will catch up to the tortoise. However, Zeno would argue that during the time Achilles was running 1 meter, the tortoise moved a little further ahead, say .09 *m*. As Achilles moves 0.9 *m* to catch up to the tortoise, the tortoise has already moved a little further along: .009 *m*. This process continues infinitely and the following series could be written to represent Achilles moves:

$$\text{Distance} = .9 + .09 + .009 + .0009 + .00009 + \cdots$$
$$= \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} + \frac{9}{100000} + \cdots$$

This is a geometric series with a common ratio of $\frac{1}{10}$. Therefore, the sum formula for infinite geometric series can be used as follow: $S = \frac{\frac{9}{10}}{1 - \frac{1}{10}} = \frac{\frac{9}{10}}{\frac{9}{10}} = 1$. Again, this series shows that if Achilles moves along this sequence of values, eventually Achilles will catch up to the tortoise.

An infinite series that diverges is one that, when all of the terms are added together, the sum will not equal a finite number. A simple example of this is the series $1 + 2 + 3 + 4 + 5 + \ldots$. Since larger and larger values are being added, the sum will tend

towards infinity. This would not be a good representation of Achilles movements because this series indicates that the distance he travels is growing larger each time. Since the assumption is that Achilles is much faster than the tortoise, it is not logical that the gap between the two would grow with each new step.

The question remains as to whether or not a diverging series exists in which the numbers in the series get smaller and smaller indicating that the gap between Achilles and the tortoise is shortening. If this type of series exists, then it could be shown that perhaps Achilles will never reach the tortoise. In fact, this type of series does exist; it is called the harmonic series [11].

$$\text{Harmonic Series} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots$$

Recall that the larger the value in the denominator, the smaller the value of the fraction, therefore each term in the series is decreasing. This is not a geometric series since each term is not multiplied by a common ratio to get the next term. Therefore, the sum formula for infinite geometric series cannot be used to find the total sum of this series. There is a clever way of determining if the series converges or diverges and calculation is shown below:

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots = \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots$$
$$\geq \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots$$
$$\geq \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

In the series to the right, $\frac{1}{2}$ is added an infinite number of times. If $\frac{1}{2}$ is added to a previous sum then the value will continue to grow towards infinity meaning that this series will converge. Since the series on the right diverges and it is less than or equal to the harmonic series, this implies that the harmonic series will diverge as well. Since the

harmonic series diverges, then Achilles will never catch the tortoise. This may not be what Zeno had in mind; however, it is a clever way to support Zeno's paradox.

Zeno's Paradox is a great way to introduce series in a Precalculus class. The story is a perfect hook to get students thinking about infinite series and walking across the classroom is a great way to demonstrate that the sum of an infinite number of terms can equal a finite value. I will use this paradox as an opener for the unit on sequences and series and circle back to it throughout the section as we discuss the difference between converging and diverging infinite series.

4. THE NUMBER *e*

The number *e* is one that is first brought to students' attention in Algebra II when studying exponential and logarithmic functions.   The number *e* is approximated as $e \approx 2.71828182846...$.   This number is an irrational number and transcendental.   An irrational number is a number that "cannot be represented by quotients of integers or by repeating or terminating decimals" [7]. A transcendental number is a real number that is not algebraic. "That is, real numbers that cannot be a root of a polynomial equation with integer coefficients" [15]. Another number that is transcendental is the famous number π.

The number *e* is fairly new to the mathematics scene in comparison to the number π. While π was approximated almost 4000 years ago, *e* was only discovered about 400 years ago. It was first "almost discovered when logarithms were invented in 1618 by John Napier" [18]. The number *e* is said to have been first discovered by the Swiss mathematician Jacob Bernoulli while he was studying compound interest [2]. Mathematician Gottfried Leibniz, in his work on calculus, identified *e* as a constant, but labeled it *b*. As with many other concepts, it was Leonhard Euler who gave the constant its modern letter designation, *e*, and discovered many of its remarkable properties" [6].

There are two ways in which the number is typically introduced. First, *e* is shown as the "natural base" for exponential functions. Before this point, students will have worked for several days on exponential functions. They will have graphed them, evaluated them and discussed the different properties. After working with exponential functions and equations, students will then apply their knowledge by solving growth and decay problems and compound interest problems. When initially solving compound interest problems, students will work with interest compounded annually, quarterly, monthly, weekly, and daily. The formula used to calculate these values is given as $A = P\left(1 + \dfrac{r}{n}\right)^{nt}$, where *A* is the amount accumulated, *P* is the initial principal, *r* is the yearly interest rate written in decimal form, *n* is the compoundings per period, and *t* is the number of periods typically given in years.

We discuss that the more often an amount is compounded, the more interest is added to the balance. Then the discussion of compounding continuously occurs. Compounding continuously means that interest is, "hypothetically, computed and added to the balance of an account every instant" [14]. "Continuous compounding is well-defined as the upper bound of 'regular' compound interest" [14]. Students are then shown the formula for compounding continuously and often remember it due to the shampoo called Pert. The formula is given as $A = Pe^{rt}$, where A is the amount accumulated, P is the initial principal, r is the interest rate written in decimal form, t is the number of periods typically given in years. Unfortunately, there is often not time in the curriculum to show students how the number e is approximated or to give them an actual definition outside of "type it into your calculator and see what value you get." When I teach, I work hard to ensure that students have a deeper understanding of what they are learning and for this reason I have started incorporating one of the definitions of the number e.

There are three definitions for the number e. Two of the definitions involve formulas students in precalculus will be able to understand. The third definition uses an integral and this is a concept students are not introduced to until calculus. The first definition is connected to the compound interest formula. Recall that the formula is $A = P\left(1 + \dfrac{r}{n}\right)^{nt}$. "Suppose that \$1 is being invested at 100% interest per year, compounded k times per year. Then the interest rate (in decimal form) is 1.00 and the interest rate per compounding period is $\dfrac{1}{k}$. According to the formula (with $P = 1$), the compounded amount at the end of 1 year will be $A = \left(1 + \dfrac{1}{k}\right)^{k}$" [7]. As stated earlier, students often work with compounding periods of annually, quarterly, monthly, weekly, and daily. The table on the next page shows the results after inputting values for k that represent these compounding periods.

Compounding Period	k	$A = \left(1 + \dfrac{1}{k}\right)^{k}$ (rounded)
Annually	1	$A = \left(1 + \dfrac{1}{1}\right)^{1} = 2$
Quarterly	4	$A = \left(1 + \dfrac{1}{4}\right)^{4} = 2.44140625$
Monthly	12	$A = \left(1 + \dfrac{1}{12}\right)^{12} = 2.61303529$
Weekly	52	$A = \left(1 + \dfrac{1}{52}\right)^{52} = 2.692596954$
Daily	365	$A = \left(1 + \dfrac{1}{365}\right)^{365} = 2.714567482$

Notice that as the number of compounding periods increases, the amount accumulated, A, gets closer in value to the number e. If this table is expanded to show larger and larger k values, the results will look like the following:

k	$A = \left(1 + \dfrac{1}{k}\right)^{k}$ (rounded)
1,000	2.716923932
10,000	2.718145927
100,000	2.718268237
1,000,000	2.718280469

In fact, as the k value tends towards infinity, the formula will tend towards the precise decimal value of the number e. This can be written as follows: $\lim_{k \to \infty} \left(1 + \frac{1}{k}\right)^k = e$.

In precalculus, limits are not covered previous to learning about exponential and logarithmic functions. However, students will understand the idea that if the value of k grows infinitely large, then the result of the formula will get closer and closer to the actual number e. One way for students to verify this is to graph the function $f(x) = \left(1 + \frac{1}{x}\right)^x$ in their graphing calculator. They will recognize that a horizontal asymptote occurs at the exact number e. They can check that this horizontal asymptote exists by looking at the table feature of their calculator. As the x-values in the table increase, the y-values will tend towards the number e. A picture of the graph is shown below.
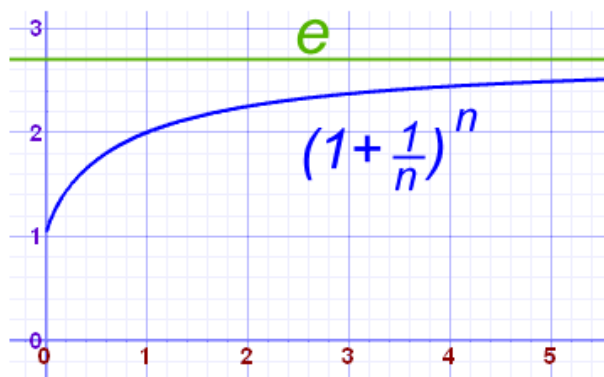
Now, students can see how the formula for compounding continuously can be obtained from the original compound interest formula. Recall that the compound interest formula is $A = P\left(1 + \frac{r}{n}\right)^{nt}$. Let $k = \frac{n}{r}$, then $rk = n \to \frac{r}{n} = \frac{1}{k}$. With these substitutions, the formula now becomes:

$$A = P\left(1 + \frac{r}{n}\right)^{nt} = P\left(1 + \frac{1}{k}\right)^{rkt} = P\left[\left(1 + \frac{1}{k}\right)^k\right]^{rt}$$

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It was shown above that $\lim\limits_{k \to \infty}\left(1+\dfrac{1}{k}\right)^{k} = e$, so that expression inside of the brackets

will approach e. Therefore, this formula will lead to $P\left[\left(1+\dfrac{1}{k}\right)^{k}\right]^{rt} \to Pe^{rt}$ as k

approaches infinity [7]. As stated earlier, given the state curriculum, there is not always time to go over the background of these formulas. However, it is important to mention them because students need to understand that formulas and equations were not just developed by magic. This exercise would be a great extension or a great way to differentiate for students who need more of a challenge.

The second way to define the number e is through the use of a series. This series was first published by Sir Isaac Newton in 1669 [20]. This definition can easily be shown when discussing infinite series. It uses the notion of infinity due to the fact that an infinite number of terms are added together to equal e. The expanded series is written below along with the equivalent summation notation.

$$e=\sum_{n=0}^{\infty}\frac{1}{n!}=\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$$

The final definition for the number e can be stated as follows: "e is the unique number with the property that the area of the region bounded by the hyperbola $y = \dfrac{1}{x}$, the x-axis, and the vertical lines $x = 1$ and $x = e$ is 1. In other words, $\displaystyle\int_{1}^{e}\frac{1}{x}\,dx=\ln e=1$" [20]. This definition is not one that precalculus students would understand; however, they will be able to understand the idea in calculus.

There are several applications for the letter e besides compounding interest continuously. Another application is found in a Bernoulli trials process. "Suppose that a gambler plays a slot machine that pays out with a probability of one in n and plays it n times. Then, for large n (such as a million) the probability that the gambler will lose every bet is (approximately) $1/e$" [2]. This means that "each time the gambler plays the

slots, there is a one in a million chance of winning. The probability of winning k times out of a million trials is" [2].

$$\binom{10^6}{k} \cdot \left(10^{-6}\right)^k \cdot \left(1-10^{-6}\right)^{10^6-k}$$

For the probability of winning zero times, $k = 0$, the expression can be given as $\left(1-\dfrac{1}{10^6}\right)^{10^6}$ which is very close to the limit for $1/e$: $\dfrac{1}{e} = \lim_{n\to\infty}\left(1-\dfrac{1}{n}\right)^n$ [2].

Another very interesting characteristic of the number e is the well-known equation $e^{i\pi} + 1 = 0$. This equation stems from the exponential function e^x written as a Taylor series where x is a real number. The expansion is shown below.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^2}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

"Because this series keep many important properties for e^x even when x is complex, it is commonly used to extend the definition of e^x to the complex numbers. This, with the Taylor series for $\sin x$ and $\cos x$, allows one to derive Euler's formula," shown below [2].

$$e^{ix} = \cos x + i\sin x$$

This equation holds for all x. The special case occurs when $x = \pi$. The simplification is shown below.

$$e^{i\pi} = \cos(\pi) + i\sin(\pi) \to e^{i\pi} = -1 + 0 \to e^{i\pi} + 1 = 0$$

This is one of the most famous equations in mathematics because it uses some of the most important numbers in mathematics: 0, 1, i, π, and e.

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In precalculus I am always looking for application problems using the number e that do not involve compounding continuously. One problem I use involves Newton's Law of Cooling. This problem was taken from Dr. Robert Kolesar's Calculus class. He shared a problem in which one has to calculate the time at which a person had died using the formula for Newton's Law of Cooling. The formula is given as follows $T(t) = T_e + (T_0 - T_e)e^{-kt}$ where $T(t)$ equals the temperature of the object at time t, T_e is the constant temperature of the environment, T_0 is the initial temperature of the object, and k is a constant that depends on the material properties of the object (k is negative because the temperature of the object is decreasing).

The number e is one that is referred to often when learning about exponential and logarithmic functions. However, students do not always understand what the number e is and how it can be calculated. Introducing these definitions in class and having the students work through them will be beneficial in their understanding of the number.

5. KOCH'S CURVE

A fractal is a never ending pattern that can occur naturally, geometrically or algebraically. These patterns are infinitely complex and "are self-similar across different scales" [21]. Fractals are created by repeating a simple process over and over again. Naturally, fractals can be seen in the branching patterns of trees, river networks, lightning bolts, blood vessels, human lungs, and spiral patterns such as seashells, hurricanes, and galaxies, just to name a few examples. Fractals can also be created by calculating a simple equation thousands of times, feeding the answer back into the start.

A famous example of a fractal is Koch's Snowflake. Koch's Snowflake was first mentioned in 1904 by mathematician Niels Fabian Helge von Koch in his paper titled "*Sur une courbe sans tangent, obtenue par une construction geometrique elementaire*". In this paper he used the Koch Curve to define the Koch Snowflake, both of which are a fractal. Using the Koch Snowflake, he showed that objects can be continuous everywhere but differentiable nowhere [4].

The Koch curve starts with a straight line segment that is divided into three equal sections. The middle section is then replaced with two sections, both equal in length to the section being replaced. Now there are four connected segments. This process is then repeated over and over again to each of the four segments. The first four iterations of this fractal are shown in the image below.

25
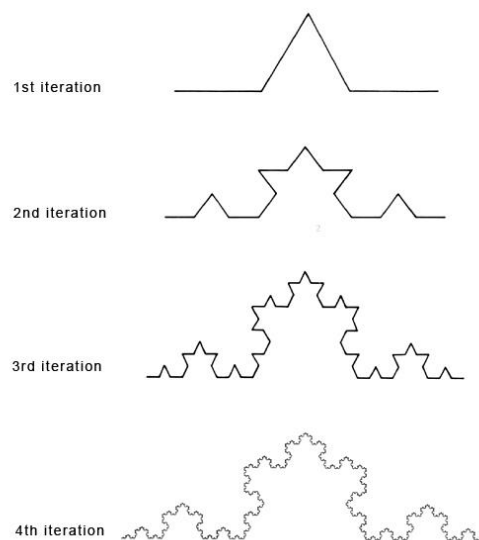
What is interesting about this curve is that when you calculate the area under the curve, you get a finite value. However, when you calculate the perimeter of the curve, you find the measurement is infinite. This means that if the iteration of the curve is completed an infinite number of times and then stretched out into a straight line, the line would be infinite and yet the area of the space under the curve is finite. At first thought, this seems impossible. It is difficult to imagine the existence of such an object because everything in the natural world is finite. However, mathematically speaking it is quite possible for the infinite to exist.

In precalculus, I want a way to challenge my students mathematically and get them to think beyond the math they are used to doing. I created an activity using Koch's Curve that allows students to use geometric series to approximate the length of the perimeter and the area under the Koch Curve if the fractal were created using infinite iterations. This activity allows students to create fractals, write geometric series, find the sums of infinite geometric series, discuss results of infinite series that converge or diverge, and be introduced to a seemingly mathematical paradox that will hopefully stretch their imagination.

In this activity, students create several iterations of the fractal by hand or using geometry software of their choice. The first stage is a single line segment. Students must start with the assumption that the length of their segment is one unit. This makes for simpler calculations and students will be able to see a pattern more easily as they find the perimeter of the following stages. Students need to be accurate in the recordings, so they are asked to represent all values as rational numbers in the table provided. On the next page is an example of a table with the results students should achieve.

Stage	Number of Segments	Length of Each Segment	Total Length (Fraction Form)
0	1	1	1
1	4	$\dfrac{1}{3}$	$4\left(\dfrac{1}{3}\right) = \dfrac{4}{3}$
2	16	$\dfrac{1}{9}$	$16\left(\dfrac{1}{9}\right) = \dfrac{16}{9}$
3	64	$\dfrac{1}{27}$	$64\left(\dfrac{1}{27}\right) = \dfrac{64}{27}$

At this point, each stage gets more difficult to create, especially by hand. The students are asked to describe any patterns they notice in the table. One pattern students should notice is that the number of segments is four times the previous number of segments for each successive stage. This can be written as 4^n where n is the stage number. Another pattern is that the length of each segment is $\frac{1}{3}$ of the previous length. This can be expressed as $\left(\frac{1}{3}\right)^n$. If they multiply the number of segments times the length of each segment, they will obtain the total length of the curve at that stage. They are to then use these patterns to fill in a table containing stages 4, 5, 6 and n without creating any more stages pictorially. For stage n, students should write an equation that would produce the total length given any stage number. In other words, the students need to come up with an explicit formula. On the next page is an example of a table with the results students should achieve.

Stage	Number of Segments	Length of Each Segment	Total Length (Fraction Form)
4	256	$\dfrac{1}{81}$	$\dfrac{256}{81}$
5	1024	$\dfrac{1}{243}$	$\dfrac{1024}{243}$
6	4096	$\dfrac{1}{729}$	$\dfrac{4096}{729}$
n	4^n	$\dfrac{1}{3^n}$	$\dfrac{4^n}{3^n} = \left(\dfrac{4}{3}\right)^n$

After discussing the explicit formula for stage n, students are asked what the length of the perimeter tends towards as the stage approaches infinity. Since the base of the exponential is greater than one, the perimeter will grow larger and larger as the n value increases to extremely large values. Students can test this by substituting larger and larger values in for n and comparing the results.

Once students have found the perimeter of the curve, they would then calculate the area under the curve. What is meant by "area under the curve" is that a straight line is drawn along the base of the curve and all of the triangles would be shaded. An example of this can be seen in the picture below. The goal is to calculate the area of the shaded region after an infinite number of iterations have been completed.
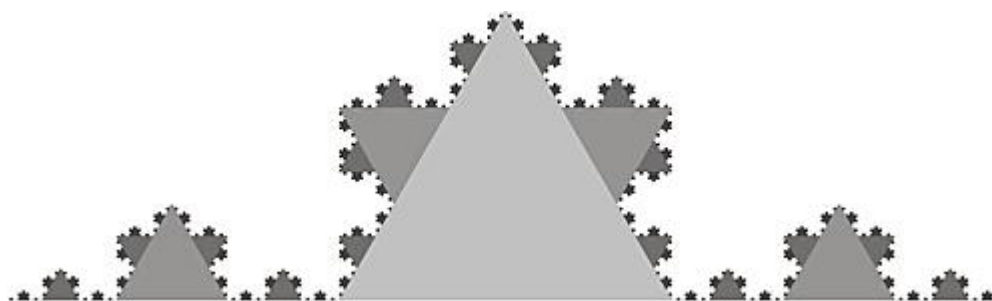
Students start off with the assumption that that area under the curve at stage 1 is A units squared. Students then complete a table asking for the area of each new triangle at the next stage and the total area. After completing the table, Students are asked to explain what patterns they notice and how they could calculate the total area as the stage

number n approaches infinity. An example of a correctly completed table is shown below.

Stage	Number of Additional Triangles	Area of Each Additional Triangle	Total Area (Fraction Form)
0	0	0	0
1	1	A	A
2	4	$\dfrac{1}{9}A$	$A+\dfrac{4}{9}A$
3	16	$\dfrac{1}{81}A$	$A+\dfrac{4}{9}A+\dfrac{16}{81}A$
4	64	$\dfrac{1}{729}A$	$A+\dfrac{4}{9}A+\dfrac{16}{81}A+\dfrac{64}{729}A$
n	4^{n-1}	$\dfrac{1}{9^{n-1}}A$	$A+\dfrac{4}{9}A+\dfrac{16}{81}A+...+\left(\dfrac{4}{9}\right)^{n-1}A$

Students should notice that to obtain the number of additional triangles, multiply the number of triangles in the previous stage by 4. The trick here is that the exponent is $n – 1$ instead of just n. The area of each additional triangle is $\frac{1}{9}$ the area of the triangles created in the previous stage. When calculating the total area, students are creating a geometric series that is equivalent to $\sum\limits_{n=1}^{\infty}\left(\frac{4}{9}\right)^{n-1}A$. Since this is a geometric series whose common ratio is between 0 and 1, the infinite series converge to a finite value. This calculation is shown below:

$$\sum_{n=1}^{\infty}\left(\tfrac{4}{9}\right)^{n-1}A=\frac{A}{1-\frac{4}{9}}=\frac{A}{\frac{5}{9}}=\frac{9}{5}A$$

The total area is $\dfrac{9}{5}$ of the area of the original triangle in stage 1. This means that the area under the Koch curve is finite when the perimeter of the curve is infinite when

the number of iterations of the curve approaches infinity! This is one of the great mathematical paradoxes and one that can be worked out by high school students. On the next page is a blank copy of the student activity sheet that students can use to work through the problem.

5A. FINDING THE PERIMETER AND AREA UNDER KOCH'S CURVE

In order to complete this activity, you must finish each part before starting the next.

I. Drawing the image:

1. On a blank sheet of paper, preferably computer paper, draw a straight line.

2. Redraw the exact same line and trisect the line.

3. Remove the middle third of the line and replace this segment with two segments whose length is congruent to the segment you removed.

4. Continue steps 1 – 3 two more times so that you are **<u>trisecting each segment</u>** in the new image. When you are finished, you should have four images.

II. Finding the Perimeter:

1. As you create your images from part I, fill in the table below (**Assume the length of the original segment is one unit**):

Stage	Number of Segments	Length of Each Segment	Total Length (Fraction Form)
1			
2			
3			
4			

2. Explain the pattern you notice as the number of segments increase.

3. Explain the pattern you notice in the length of each segment as the stage number increase.

4. Using your answers to #2 and #3 directly above, complete the following table without drawing more images.

Stage	Number of Segments	Length of Each Segment	Total Length (Fraction Form)
4			
5			
6			
n			

5. Write a function to find the perimeter of the *n*th stage:

6. As the stage approaches infinity, what does the perimeter tend towards? Explain.

III. Finding the Area:

1. Let's find the area underneath this curve. By area "underneath" I mean finding the area under the triangles created as the stage moves towards infinity.

2. To make things a little simpler, we are going to assume that the area under the equilateral triangle in stage 1 is given as A.

3. Complete the following chart:

Stage	Number of Additional Triangles	Area of Each Additional Triangle	Total Area (Fraction Form)
0	0	0	0
1	1	A	A
2			
3			
4			
n			

4. Explain the pattern you notice as the number of additional triangles increase.

5. Explain the pattern you notice as we start to sum the total area.

6. As the stage approaches infinity, what does the area tend towards? Explain

Reflect: What did you find out about this curve that seems "unnatural"?

6. GABRIEL'S HORN

The painter's paradox is a famous paradox that claims a three-dimensional figure, resembling a horn, that is created by revolving the function $f(x) = \dfrac{1}{x}$ about the x-axis from one to infinity, has finite volume but infinite surface area. The story says that "since the Horn has finite volume but infinite surface area, it seems that it could be filled with a finite quantity of paint, and yet the paint would not be sufficient to coat its inner surface – an apparent paradox" [5]. This paradox is often referred to as Gabriel's Horn and was named so for the Archangel Gabriel who is known "as the angel who blows the horn to announce Judgement Day, associating with the divine, or infinite, with the finite" [5]. The properties of this figure were discovered by the Italian physicist and mathematician Evangelista Torricelli in 1641. Torricelli was the successor to Galileo at Florence [3]. As a result, the paradox is sometimes referred to as Torricelli's Trumpet.

Torricelli did not have the advantage of calculus to calculate the surface area and volume of this curve. Instead, he used Cavalieri's Principle. Cavalieri's Principle for the volume of three-dimensional objects states that "Suppose two regions in three-space (solids) are included between two parallel planes. If every plane parallel to these two planes intersects both regions in cross-sections of equal area, then the two regions have equal volumes" [10]. This means that if the two objects have the exact same height and all corresponding cross-sections have the exact same area, then the objects are equal in volume. An example of this is shown in the diagram below.
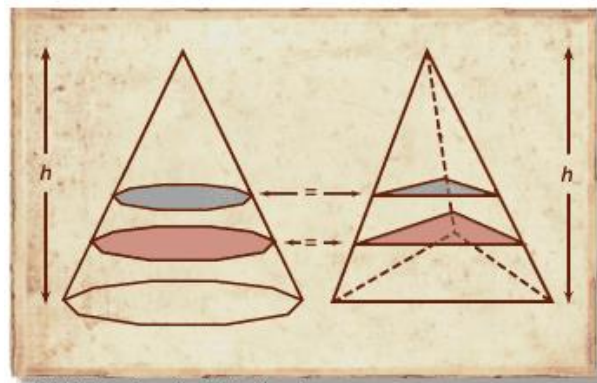
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However, today calculus can be used to find the surface area and volume of this solid. This activity is something students in second semester calculus could compute after discussion of rotating a curve around the x-axis. First, students will graph the function $f(x) = \dfrac{1}{x}$ on a coordinate plane. Then, students will graph the rotation of the curve about the x-axis. The image of this graph is shown below.
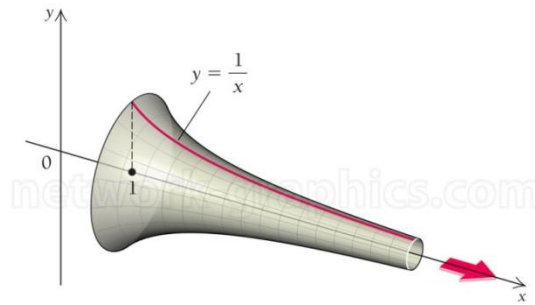
After graphing the rotation of the graph of the function $f(x) = \dfrac{1}{x}$, students will then find the volume inside the object on the interval $[1, \infty)$. The formula to the find the volume of an object rotated around the x-axis is $V = \int_1^\infty \pi \left(f(x) \right)^2 dx$ (Source 14). Since the upper bound is infinity, students will eventually need to take a limit of the results of the integrated function. When calculating this volume for the given function of $f(x) = \dfrac{1}{x}$, students will find that the result is π, which is a finite number since pi is approximately 3.14159. Below is the calculation the students should derive to obtain the finite answer.

$$V = \int_1^\infty \pi \left(f(x) \right)^2 dx$$

$$= \int_1^\infty \pi \left(\frac{1}{x} \right)^2 dx = \lim_{n \to \infty} \int_1^n \frac{\pi}{x^2} dx = \lim_{n \to \infty} \left(-\frac{\pi}{x} \right) \Big|_1^n = \lim_{n \to \infty} \left(-\frac{\pi}{n} + \pi \right) = \pi$$

Students will then find the surface area of the curve on the interval $[1, \infty)$. The formula to the find the surface area of an object rotated around the x-axis is $SA = \int_1^{\infty} 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$ [3]. Since the upper bound is infinity, students will eventually need to take a limit of the results of the integrated function just like they did when finding the volume. When calculating the surface area for the given function of $f(x) = \dfrac{1}{x}$, students will find that the result is infinite. This means that there is no finite answer for surface area. Below is the calculation the students should derive to obtain the finite answer.

$$SA = \int_1^{\infty} 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$
$$= \int_1^{\infty} 2\pi\left(\frac{1}{x}\right)\sqrt{1+\left(-\frac{1}{x^2}\right)^2}\,dx = 2\pi\int_1^{\infty}\left(\frac{1}{x}\right)\sqrt{\frac{x^4+1}{x^4}}\,dx = 2\pi\int_1^{\infty}\left(\frac{1}{x^3}\right)\sqrt{x^4+1}\,dx$$

This last integral is difficult to calculate, so the following true statement is made in order to make the calculation easier:

$$2\pi\int_1^{\infty}\left(\frac{1}{x^3}\right)\sqrt{x^4+1}\,dx \;\geq\; 2\pi\int_1^{\infty}\left(\frac{1}{x^3}\right)\sqrt{x^4}\,dx$$

From here, the students can proceed with the following calculations:

$$SA = 2\pi\int_1^{\infty}\left(\frac{1}{x^3}\right)\sqrt{x^4+1}\,dx \;\geq\; 2\pi\int_1^{\infty}\left(\frac{1}{x^3}\right)\sqrt{x^4}\,dx$$
$$\downarrow \qquad\qquad \geq \qquad\qquad \downarrow$$
$$SA = \lim_{n\to\infty} 2\pi\int_1^{n}\left(\frac{1}{x^3}\right)\sqrt{x^4+1}\,dx \geq \lim_{n\to\infty} 2\pi\int_1^{n}\left(\frac{1}{x}\right)dx = 2\pi\lim_{n\to\infty}\left.(\ln x)\right|_1^{n}$$
$$= 2\pi\lim_{n\to\infty}\left(\ln n - \ln 1\right)$$
$$= \infty$$

Since there is finite volume but infinite surface area, a paradox is created mathematically. This level of math is much too difficult for first year calculus students. Therefore, I found a more approachable way of finding the same conclusions called Gabriel's Wedding Cake [3]. In this activity, students will graph and revolve a function around the x-axis, use a step-function to represent the function, calculation the volume and surface area of the newly defined step-function, write the volume and surface areas as infinite series and then calculate the sum of the infinite series. This is a great activity to assess students' understanding of infinite series and of functions.

Students are first asked to write the function $f(x) = \dfrac{1}{x}$. The function should be written as follows:

$$f(x) = \begin{cases} 1, & 1 \le x < 2 \\ \frac{1}{2}, & 2 \le x < 3 \\ \frac{1}{3}, & 3 \le x < 4 \\ \vdots & \vdots \\ \frac{1}{n}, & n \le x < n+1 \end{cases}$$

After writing the function, students will then graph the step-function. Below is an image of what the graph will look. It looks similar to a cake with an infinite number of layers, hence the reason for the name Gabriel's Wedding Cake.
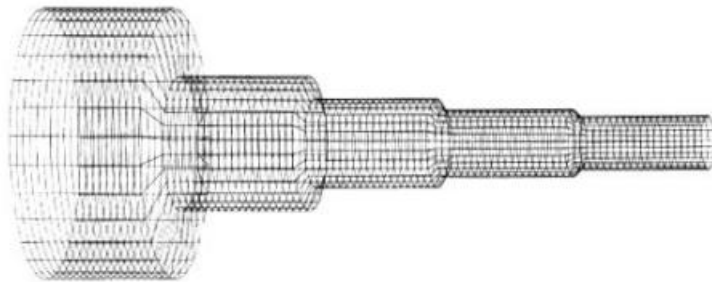
To find the volume of each layer, students will recall that the volume of a cylinder is $V = \pi r^2 h$ where r is the function $\dfrac{1}{n}$ for each step n of the step-function. Students will be finding the sum of the volumes of every layer and they can represent this sum as $V = \sum\limits_{n=1}^{\infty} \pi \left(\dfrac{1}{n}\right)^2 = \pi \sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$. This series is called the reciprocals of squares and can be expanded to equal:

$$\pi \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \cdots\right) = \pi\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \cdots\right)$$

The mathematician Leonhard Euler solved this in 1735 and using the expansion of $\dfrac{\sin(x)}{x}$. Euler proved that the sum of the series is equal to $\dfrac{\pi^2}{6}$, therefore, the volume of Gabriel's Wedding Cake would be equal to $\pi \cdot \dfrac{\pi^2}{6} = \dfrac{\pi^3}{6} \approx 5.1677$ [1]. However, this may prove to be too difficult for first year Calculus students. In order to approximate the solution and show that this series converges, students will need to write a series that is greater than or equal to the summation of the reciprocals of squares. The following inequality statement could be made [3]:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64}\cdots \leq 1 + \frac{1}{4} + \frac{1}{4} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{64}\cdots$$

$$\leq 1 + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\right) + \frac{1}{4}\left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}\right) + \frac{1}{8}\left(\frac{1}{8} + \cdots\right) + \cdots$$

$$\leq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

This last series is the geometric series which converges to the finite value of 2. Therefore, if the series on the right, the geometric series, converges and the series on the left is less than the geometric series, then the series on the left also converges. This means that the volume of Gabriel's Wedding Cake could be approximated to be less than

$2\pi \approx 6.2832$. This is true compared to Euler's solution. Although this method does not give an exact solution, it is approachable for first year Calculus students and is a great application of comparing series and geometric series.

Students will then calculate the surface area of the step-function. Students will recall that to find the surface area of a cylinder they must know the area of the base, a circle, and the area of the lateral side. If the Wedding Cake were collapsed so that all of the layers were nested inside each other, then the result would be a disk with a radius of one [3]. Therefore, the total surface area of the top of the disk is $\pi(1)^2 = \pi$. The lateral sides can be found using the formula $2\pi rh$ where r is the step-function at each interval and h is the width of each section which is one. Therefore, the total surface area of the lateral sides can be given as the sum $A_L = \sum_{n=1}^{\infty} 2\pi\left(\frac{1}{n}\right)(1) = 2\pi\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)$ [4]. This last summation is a representation of the harmonic series. When expanded the series is given as $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$. Students need to prove that this series is diverging in order to show that the surface area is infinite. This was proven earlier in the paper when showing that Zeno's Paradox mathematically be represented. Below is the calculation students will need to show to prove the harmonic series diverges.

$$
\begin{aligned}
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots \\
&\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots \\
&\geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots
\end{aligned}
$$

The final series diverges since $\frac{1}{2}$ is continually being added. The series will tend towards infinity. Students will have to finish the calculation by multiplying infinity by 2π. The product is infinity. This now represents the total surface area of the lateral sides. Finally, students will add the total surface area of the tops of the cylinders, which is π. Again, infinity plus π is still equal to infinity. Therefore, students will show that

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this function, after being rotated around the *x*-axis, has finite volume and infinite surface area. This may not have been the way that Evangelista Torricelli determine the solutions, however, this method exposes students to two of the most important series in mathematics; the harmonic series and the geometric series. On the next page is a blank copy of the student activity sheet that students can use to work through the problem.

6A. FINDING THE VOLUME AND SURFACE AREA OF GABRIEL'S HORN

In order to complete this activity, you must finish each part before starting the next.

I. Graphing the Function:

1. On a coordinate plane, graph the function $f(x) = \dfrac{1}{x}$.

2. State three characteristics that you notice about the graph of the function.

   a.

   b.

   c.

3. Sketch the graph of the function rotated about the x-axis on the interval $[1, \infty)$. What does this image look like?

4. Draw a second graph to represent the given step-function. Then rotate the function about the x-axis on the interval $[1, \infty)$. What does this image look like?

$$f(x) = \begin{cases} 1, & 1 \le x < 2 \\ \frac{1}{2}, & 2 \le x < 3 \\ \frac{1}{3}, & 3 \le x < 4 \\ \vdots & \vdots \\ \frac{1}{n}, & n \le x < n+1 \end{cases}$$

II. Finding the Volume:

1. Recall the formula for finding the volume of a cylinder.

2. Represent the total volume of the rotated function using summation notation and as an expanded series.

3. Write a series that is very common and is greater than or equal to the expanded series in #2. Prove or explain why the series you come up with is greater than or equal to the expanded series in #2.

4. Find the sum of the new series. Does it converge or diverge? What does this mean for the initial series? What does this mean about the volume of the function rotated about the *x*-axis on the interval $[1, \infty)$?

III. Finding the Surface Area:

1. Recall the formula for finding the surface area of a cylinder.

2. Since the function is open on one side, you only need to be concerned about finding the area of one base. If the function were to "collapse" on itself so that all of the layers were together, then the result would be a disk with a radius of what length? What is the area of the disk?

3. Represent the total surface area of the lateral sides of the rotated function using summation notation and as an expanded series.

4. Write a series that is very common and is less than or equal to the expanded series in #3. Prove or explain why the series you come up with is less than or equal to the expanded series in #3.

5. Find the sum of the new series. Does it converge or diverge? What does this mean for the initial series? What does this mean about the surface area of the lateral sides of the function rotated about the x-axis on the interval $[1, \infty)$?

6. Use your results from #2 and #5 to find the total surface area of the function rotated about the x-axis on the interval $[1, \infty)$.

Reflect: What did you find out about this curve that seems "unnatural"?

7. CONCLUSION

The goal of my paper is to create lessons or discussions that will introduce students to the concept of infinity and see how it plays into the mathematics that they do. The two lessons that precalculus students might find the most interesting and will find the most difficult are Koch's curve and Gabriel's horn. These two are written as activities so that students are exposed to a rich mathematical problem with results that seem impossible. These two lessons incorporate patterns and measurements that can be written as infinite series. Students must then find the sums of those infinite series or prove that the series either converges or diverges by comparing their series to another one in which the sum is known. These problems test that students understanding of what it means to be infinite both mathematically and pictorially. These two problems represent some of my favorite mathematical thinking since you have to be creative in your thought process and pull on previous knowledge. When these lessons are finished, it will be important to discuss with students that while the mathematical results show that perimeter or surface area are infinite, in the natural world, it is not possible for either to be infinite.

The other three topics do not have specific lesson activities created because they pose great discussion points that are intertwined with sections of mathematics. As an introduction to algebra I, students review the number systems. The discussion and proofs that $0.999\ldots = 1$ is a great discussion to have within the context of that lesson. Students are usually satisfied that $\frac{1}{3} = 0.333...$, however it is difficult for some to accept that $0.999\ldots = 1$. The assumption is that eventually the number of nines will stop and therefore it is not possible for $0.999\ldots$ to be equal to one. The different proofs given in the section will give students a better grasp on this truth using varying levels of mathematics. This idea could also be used as an extension for students who want more of a challenge or understand number systems faster than others.

As mentioned earlier, Zeno's paradox is a fantastic way to introduce infinite series to my precalculus class. Students have a hard time understanding that they can add up an infinite number of terms and obtain a finite sum. This may be the first time they truly feel the struggle of understanding infinity. Up until this point, their assumption was

that if you keep adding more and more numbers, your sum will continue to grow larger and larger and will not stop at a finite number.

The third topic, the number *e*, is important as students are usually not given the formal mathematical definition of *e*. They are simply told that it is a natural number and used as a base for exponential and logarithmic functions and equations. Again, I did not create a separate activity for this section because the definition of the number *e* is a great discussion to have within the context of the section on exponential functions.

Overall, I know that having these discussions or completing the activities in my classes throughout the year will give students a better understanding the mathematics they are doing and help them feel more comfortable with the concept of infinity. These discussion and lessons will allow for deep mathematical thought, problem solving and critical thinking. Students will have the opportunity to have mathematical discussions that help them to become more independent thinkers and learners. The previous two statements are the foundation of how I teach because I want the learning to begin with the students. This experience has allowed me to develop a better appreciation for how often the concept of infinity is used in the mathematics that I teach and gives me a great resource as I move throughout the year.

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