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Hyperbolic Geometry and Mobius Transformations

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HYPERBOLIC GEOMETRY AND MÖBIUS TRANSFORMATIONS

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INTRODUCTION

Imagine a world in which there are infinitely many lines through a single point that are all parallel to the same line; this world is referred to as the hyperbolic plane. This paper is an introduction to the notions of the hyperbolic plane. It starts with a brief history of the development of hyperbolic geometry and the mathematicians who contributed their explorations of the hyperbolic plane. The reader will be reminded of some of the important aspects of Euclidean geometry and some of the basic properties that will later be adjusted to develop hyperbolic geometry. The hyperbolic plane will be constructed from the pseudosphere and the properties of the Poincaré Upper Half plane will be explored. The theory of Möbius transformations and their properties will be developed leading to an examination of hyperbolic isometries, which will be used to construct other models of the hyperbolic plane.

CHAPTER 1: HISTORY

Geometry has been determined to have originated in Ancient Egypt; where geometry consisted of isolated facts of observations and simple rules for calculations. Thales of Miletus introduced a more modern treatment of geometry to Greece and from there geometry as we know it today began. Democritus was a fifth-century B.C. Greek philosopher who boasted that no one could surpass him in his knowledge of geometry, not even the Egyptian Harpenodapts; giving insight that in his time Egyptians were thought to be the most skillful geometers. Many believe that Democritus knew more than what is taught in today's high schools. Next came Eudoxus, who is given credit for formally organizing the theorems of geometry into a structure that begins with axioms and proceeds to derive theorems in a systematic matter. His books and writings did not survive over the years. What we know about him is through second hand knowledge. Hippocrates of Chios also attempted to organize geometry but the most famous of such attempts was that of Euclid.

Euclid's famous books, *The Elements* circa 300 B.C., have been described as an incomparable masterpiece of systematic, deductive Greek thought. *The Elements,* which consists of 13 books, has been translated into many languages and defined the content of geometry for many cultures. Our most immediate concern is Book I where Euclid describes the foundations of geometry. Euclid states five postulates that are exclusively geometrical. The first four were accepted as postulates and are the axiomatic basis of what is known as absolute geometry. They are: 1. To draw a straight line from any point to any point, 2. To produce a finite straight line continuously in a straight line, 3. To describe a circle with any center and distance, 4. All right angles are equal to one another. The fifth postulate is also called the parallel postulate and was believed to be unnecessary. It was thought that its validity could be demonstrated on the basis of the first four postulates. The fifth postulate states that if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, will meet on that side where the angles are less than the two right angles. Euclid first used the parallel postulate to prove that the alternate interior angles formed by two parallel lines cut by a transversal must be equal.

Over two millennia, mathematicians have repeatedly attempted to prove the parallel postulate using the first four postulates. The first documentation of the fifth postulate being questioned was by Proclus. He wrote a commentary on Euclid's Book I and proceeded to prove the fifth postulate. His attempt was flawed because he assumed that two parallel straight lines are at a constant distance from one another which is a statement logically equivalent to the fifth postulate. From then on there were repeated attempts made by Arab mathematicians during the Middle Ages and by several European mathematicians during the Renaissance, none of which proved to be valid. Each successive demonstrator showed the falseness of his predecessor's reasoning or pointed out an assumption used that was considered to be equivalent to the parallel postulate. Some of these theorems include: the sum of the angles of a triangle is equal to two right angles, Playfair's postulate which states that given a line and a point not on the line, there exists exactly one line through the given point that is parallel to the given line, and parallel segments contained between two parallel straight lines are equal. Thus, there was no advancement made toward a settlement of the question, could the parallel postulate be proven using the first four postulates?

Then, in 1667, a Jesuit, Gerolamo Saccheri, devised an entirely different approach to this problem. He decided to do a proof by contradiction. He started with two equal perpendiculars AC and BD to a line AB. He showed that when the endpoints C and D are joined, the two angles at C and at D are equal. Saccheri keeps an open mind and proposes three hypotheses: 1. They are right angles, 2. They are obtuse angles, 3. They are acute angles. From there, he planned to demolish the last two options, thus leaving the first. His reasoning contained an error and there really was not a contradiction where he thought he saw one. He established several theorems along his journey to the contradiction. He eventually believed to have finished his proof but did not seem to be satisfied with the validity. He offered another attempt but lost himself in the quicksand of the infinitesimal. He had no faith in the negation of the fifth but if he would have had a little more imagination and been less bound by tradition, he would have anticipated the discovery of non-Euclidean geometry from the third hypothesis.

Fifty five years later, J.H. Lambert carried out the valid portions of Saccheri's work and derived many more theorems. He established the formulas for the areas of triangles in both hyperbolic and elliptic geometries. He dismissed the hypotheses of the obtuse angle because it required two straight lines to enclose a space but he was tempted to draw the conclusion that the third hypothesis (hyperbolic geometry) arises with an imaginary spherical surface. He ignored this thought and continued to demonstrate what he concluded to be the validity of the parallel postulate. Like Saccheri, Lambert arrived at a conclusion that strongly contradicted his observations of his physical universe. He could not help but conclude that he had arrived at a logical inconsistency. His research was not published until years after his death.

About the same time, Gauss was attracted to the same question. He only published a few reviews but it was clear that he was very interested in the subject. He was the first to entertain serious doubts about the demonstrability of the fifth postulate and to conceive a valid non-Euclidean geometry. We attribute the name of non-Euclidean geometry to Gauss but many of his contributions were not well documented and thus not as well known.

About 1815, a professor of mathematics at Kazan, Nikolai Ivanocich Lobachevsky became interested in the theory of parallels. He wrote an article titled "On the Principles of Geometry", in which he explains the principle of his "Imaginary Geometry". He assumed that given line m and a point p not on m, there exist more than one line through p that are parallel to m. From this point he proceeds on to develop hyperbolic geometry in a synthetic manner.

In 1868, Beltrami was the first to complete the proof of the relative logical consistency of the hyperbolic plane. He pointed out that the trigonometry of the geodesics of the pseudosphere was identical to the trigonometry of the hyperbolic plane. Consequently, any self-contradiction that might arise in hyperbolic geometry would also constitute a self-contradiction of Euclidean geometry. In other words, he proved that hyperbolic geometry was at least as consistent as Euclidean geometry. Beltrami explicitly formulated the Riemannian metrics that define the upper half-plane, for the unit disk model and the Beltrami-Klein model. Of all of these great mathematicians, it never occurred to them that the sum of the angles of a triangle may be greater than two right angles. Bernhard Riemann deduced a straight line could be unbounded but of finite length; he thought somewhere the two ends of the line would meet and enclose it. He explained that two straight lines intersect twice, like two great circles on a sphere in his Dissertation of 1854. Felix Klein saw a geometry in which the straight line is finite and is uniquely determined by two distinct points. Klein called the geometry of Lobachevsky *Hyperbolic* geometry, the geometry of Riemann *Elliptic*, and the geometry of Euclid *Parabolic*.

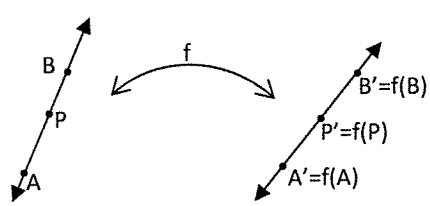
CHAPTER 2: ISOMETRIES OF THE EUCLIDEAN PLANE

We will start out with the trace of a curve with a direction along it. Consider the directed trace of a curve as a rigid object in $\mathbb{R}^n$ and move it around in space by a motion that preserves rigid bodies. For example, we can move a curve to the left two units and up 3 units. The properties of curves that remain unchanged by such motions are called geometric properties. To clarify the idea of a rigid motions, consider the following definition. A mapping $f:\mathbb{R}^n \to \mathbb{R}^n$ is an *isometry* if, for all $\vec{v},\vec{w} \in \mathbb{R}^n$, $\left\| f(\vec{v}) - f(\vec{w}) \right\| = \left\| \vec{v} - \vec{w} \right\|$. With this definition, we have translated the idea of rigidity into the idea of distance preserving mappings.

For the rest of the section, we will consider isometries from $\mathbb{R}^2$ to $\mathbb{R}^2$. So we are looking at transformations $f(P)$ of the plane into itself such that $d(P',Q') = d(P,Q)$ where $P' = f(P)$ and $Q' = f(Q)$. We define an identity transformation to be the transformation, Id, that carries every point of the plane onto itself. Note that the identity transformation is an isometry.

Proposition 2.1: *Every isometry transforms straight lines into straight lines.*

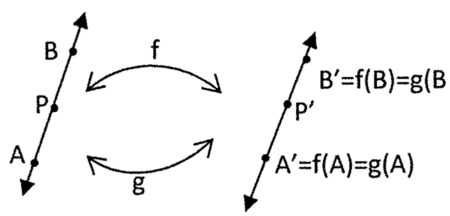
Proof:



Let AB be a line and let P be on the line AB. Without loss of generality, assume that the point P lies between the points A and B. Since A, B, and P are collinear, $d(A,B) = d(A,P) + d(P,B)$. Let $f(P)$ be an isometry. Then by the definition of an isometry, $d(A,B) = d(A',B')$, $d(A,P) = d(A',P')$, and $d(P,B)=d(P',B')$. Therefore $d(A',B')=d(A',P')+d(P',B')$ and hence P' is on the line $A'B'$. So every isometry transforms straight lines into straight lines. ■

Proposition 2.2: *If two isometries agree on two distinct points, then they agree everywhere on the straight line joining those two points.*

Proof:



Let f and g be isometries and let AB be a line such that $f(A)=g(A)$ and $f(B)=g(B)$. Also let P be on the line AB and without loss of generality assume P is between A and B. Since A, B, and P are collinear, $d(A,B)=d(A,P)+d(P,B)$.

From Proposition 2.1, $d(f(A),f(B))=d(f(A),f(P))+d(f(P),f(B))$ and $d(g(A),g(B))=d(g(A),g(P))+d(g(P),g(B))$. Since $f(AB)=g(AB)=A'B'$,

then $d(f(A),f(P))+d(f(P),f(B))=d(g(A),g(P))+d(g(P),g(B))$.

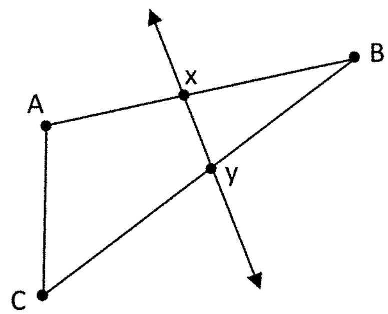
Since $f(A)=g(A)$ and $f(B)=g(B)$, then

$$d(g(A),f(P))+d(f(P),g(B))=d(g(A),g(P))+d(g(P),g(B)).$$

Since $d(f(P),g(B))=d(P,B)=d(g(P),g(B))$, then $f(P)=g(P)$. ■

Theorem 2.3: *If two isometries agree at three noncollinear points, then they agree everywhere.*

Proof:



Let f and g be two rigid motions and let A, B, and C form a triangle such that $f(A)=g(A)$, $f(B)=g(B)$, and $f(C)=g(C)$. Then we can draw a line that intersects $\triangle ABC$ at two points, X and Y. Then from proposition 2.2 $f(X)=g(X)$ and $f(Y)=g(Y)$. Thus by proposition 2.2, for any point P, f and g will agree. ■

So now that we know that three points determine an isometry, then the following corollary becomes evident.

Corollary 2.4: *If an isometry fixes three noncollinear points, then it must be the identity mapping.*

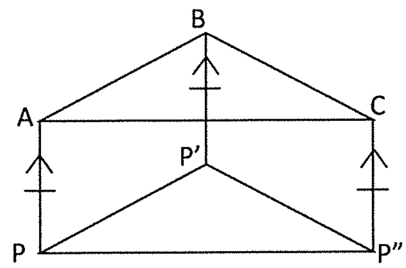
Now we are going to classify isometries of the Euclidean plane into translations, rotations, reflections, and glide reflections.

The first type of isometry is called a *translation.* We will denote a translation by τ and is define it as an isometry such that the line segments PP' and QQ' have the same length and direction whenever $P'=\tau(P)$ and $Q'=\tau(Q)$. Think of a translation as sliding throughout the Euclidean plane, and thus a translation creates a parallelogram.

We will consider the identity translation to be the trivial translation, in which the curve stays in place.

Proposition 2.5: *If A, B, and C are any three points then* $\tau_{BC} \circ \tau_{AB} = \tau_{AC}$.

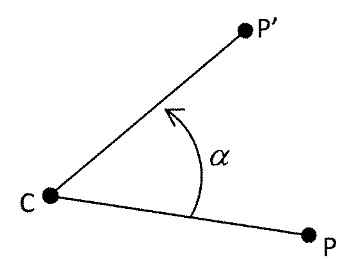
Proof:



Let τ_{AB} and τ_{BC} be translations. Let P be an arbitrary point. Then by the definition of a translation $\tau_{AB}(P)=P'$, $ABP'P$ is a parallelogram which implies $AP \parallel BP'$ and $AP \cong BP'$. Similarly, $\tau_{BC}(P')=P''$ and thus $BCP''P'$ is a parallelogram which implies $BP' \parallel CP''$ and $BP' \cong CP''$. It follows that $AP \parallel CP''$ and $AP \cong CP''$, and hence $ACPP''$ is a parallelogram. So $AC \parallel PP''$ and $AC = PP''$. Therefore $\tau_{AC}(P)=P''$. So $\tau_{BC} \circ \tau_{AB} = \tau_{AC}$. ■

It can easily be shown from proposition 2.5 that $\tau_{AB} \circ \tau_{BA} = Id$ and $\tau_{BA} \circ \tau_{AB} = Id$. Thus $(\tau_{AB})^{-1} = \tau_{BA}$.

Before defining the next type of isometry, we need the following definition. An *oriented angle* is an angle together with an orientation either clockwise or counterclockwise. All positive angles are assumed to have a counterclockwise orientation and all negative angles are assumed to have a clockwise orientation.

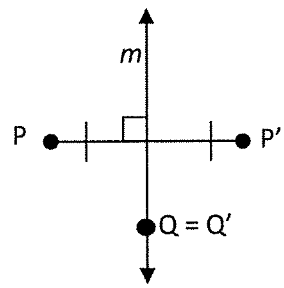


So let C be a given point and let α be a given oriented angle. The next type of isometry is called a rotation. A *rotation*, $R_{C,\alpha}$ is the function that associates to any point P the unique point P' such that

$$CP = CP' \text{ and } \angle PCP' = \alpha .$$

It can be easily shown from the definition that $R_{C,-\alpha} \circ R_{C,\alpha} = Id$ and $R_{C,\alpha} \circ R_{C,-\alpha} = Id$. Therefore the inverse of $R_{C,\alpha}$ is $R_{C,-\alpha}$.

A third type of isometry is known as a reflection. Given a straight line *m*, the *reflection* ρ_m is the transformation that fixes every point on *m* and that associates to each point *P* not on *m* the unique point $P' = \rho_m(P)$ such that *m* is the perpendicular bisector of the line segment PP'.



It can easily be seen that each reflection is its own inverse.

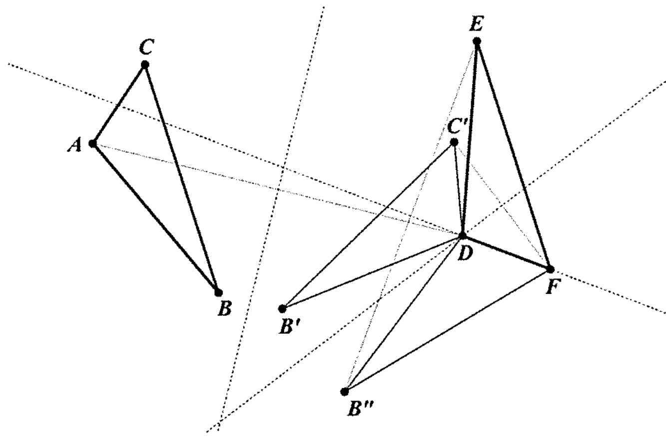
A forth type of rigid motion is called a glide reflection. Let A and B be two distinct points. The composition $\rho_{AB} \circ \tau_{AB}$ is called a *glide reflection* and is denoted by γ_{AB}.

A glide reflection is a composition of a translation and a reflection, making it clear the inverse of γ_{AB} is γ_{BA}.

The composition of the identity translation and a reflection, is just a reflection. Therefore a reflection is thought of as a special type of glide reflection.

Proposition 2.6: *Suppose* $\triangle ABC \cong \triangle DEF$. *Then, there exists a sequence of no more than three reflections such that the composition of these reflections maps the points A, B, and C, onto D, E, and F, respectively.*

Proof: Suppose $\triangle ABC \cong \triangle DEF$



Clearly there is a reflection ρ_1 that will map point A to point D. Thus $\triangle ABC$ will map to $\triangle DB'C'$ where $B' = \rho_1(B)$ and $C' = \rho_1(C)$. Note that $DC' = AC = DF$, and therefore the line through D and perpendicular to FC' is the perpendicular bisector of FC'. Thus there exists a reflection ρ_2 that will map C' to F and fix D. So $\triangle DB'C'$ is mapped onto $\triangle DB''F$ where $B'' = \rho_2(B')$. Finally, note that $DB'' = DE = AB$ and so the line through D and F is the perpendicular bisector of EB''. Then there exists a reflection ρ_3 that will map $\triangle DB''F$ onto $\triangle DEF$. Therefore the map $f = \rho_3 \circ \rho_2 \circ \rho_1$ does indeed map the points A, B, and C onto the points D, E, and F respectively. ■

Theorem 2.7: *Every isometry is the composition of at most three reflections.*

Proof: Let f be an isometry that maps three noncollinear points to the image of each point. Note that three reflections will also map the three noncollinear points to the same image by Proposition 2.6 above. Let $g = \rho_3 \circ \rho_2 \circ \rho_1$. Then f and g agree on three noncollinear points and hence, by Proposition 2.3 they agree on every point. So every isometry is the composition of at most three reflections. ■

Theorem 2.8: *Every isometry is a translation, rotation, or a glide reflection.*

Proof: Let f be an isometry. Then f is the composition of at most three reflections. If f is the composition of zero reflections then it is the identity map. If f is one reflection, then clearly it is a reflection. It can be shown that the composition of two reflections is a rotation (if the lines of reflection intersect) or a translation (if the lines of reflection are parallel). It can also be shown that the composition of three reflections is a glide reflection. Therefore the only isometries are translation, rotation, or a glide reflection. ■

With the existence of an identity mapping and an inverse for each isometry, as well as the idea that composition of isometries is an isometry and function composition is associative, the following theorem follows directly.

Theorem 2.9:*The set of all isometries form a group.*

CHAPTER 3: ISOMETRIES OF THE COMPLEX PLANE

Recall that $z = x + iy$ is a complex number where $x, y \in \mathbb{R}$ and $i = \sqrt{-1}$. The complex numbers enjoy the same basic operations of real numbers: addition, subtraction, multiplication, and division. So let $c = a + bi$ be a fixed complex number and let $z = x + yi$ be an arbitrary complex number. Then $z + c = (x+a) + (y+b)i$ and clearly the line segment from z to $z+c$ is parallel to the line segment from O to C. So if c is any fixed complex number, then the function $f(z) = z + c$ is a *translation* of the Euclidean plane.

We will also define $e^{i\theta} = \cos\theta + i\sin\theta$ to be the complex number of modulus 1 whose argument is θ and so $re^{i\theta}$ denotes the complex number of modulus r whose argument is θ. Also note $e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$. From here we have some properties of complex numbers. The product rule is denoted by $zw = r_1e^{i\theta}r_2e^{i\phi} = r_1r_2e^{i(\theta+\phi)}$ for $z = r_1e^{i\theta}, w = r_2e^{i\phi} \in \mathbb{C}$. Similarly the quotient rule is denoted by $\frac{z}{w} = \frac{r_1e^{i\theta}}{r_2e^{i\phi}} = \frac{r_1}{r_2}e^{i(\theta-\phi)}$ for $z = r_1e^{i\theta}, w = r_2e^{i\phi} \in \mathbb{C}$. Thus let α be a fixed angle and $z = re^{i\theta}$ be an arbitrary complex number. Then clearly $e^{i\alpha}z$ is a rotation of the complex plane about the origin. So if we translate a point z by $-c$, rotate about the origin by α and then translate it back by c we get a rotation about a given point c. From this idea, we can define $f(z) = e^{i\alpha}(z-c) + c = e^{i\alpha}z + (1-e^{i\alpha})c$ to be the *rotation* of an angle α about the point c, denoted $R_{c,\alpha}$.

If $z = x + iy$ then $\overline{z} = x - iy$ is the *conjugate* of z. Note that a point and its conjugate are symmetric to the x-axis and can be thought of as a reflection about the x-axis. Note the following properties: $\overline{e^{i\theta}} = e^{-i\theta}$, $\overline{z \pm w} = \overline{z} \pm \overline{w}$, $\overline{zw} = \overline{z} \cdot \overline{w}$, and $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$. Also note $\arg(\overline{z}) = -\arg(z)$ and $|\overline{z}| = |z|$. If m is a line through the origin and θ is the angle that

m makes with the positive x-axis, then the reflection ρ_m is denoted by $R_{0,\theta} \circ \rho_x \circ R_{0,-\theta}$. We can write the reflection about a line through the origin as $e^{i\theta}\overline{e^{-i\theta}z} = e^{2i\theta}\overline{z}$ in the complex plane. So if we extend this to any line we can denote the *reflection* about a given line m by $f(z) = e^{2i\theta}\left(\overline{z-c}\right)+c$ where c is a point on the line m.

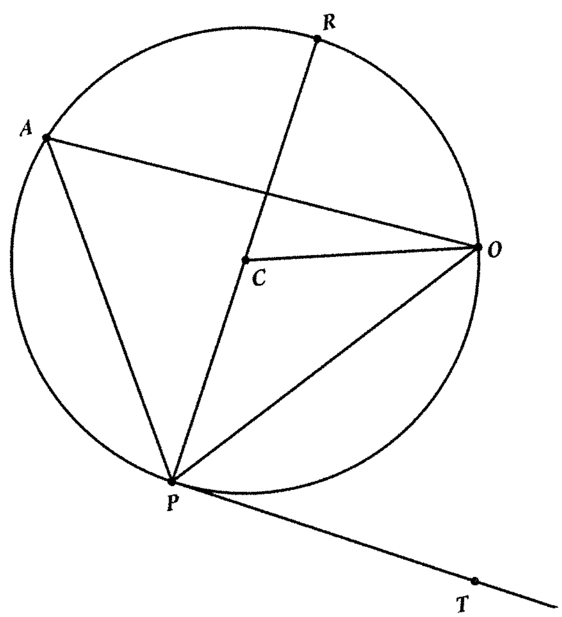
We can summarize the results of translations, rotations, and reflections in the complex plane. The isometries of the Euclidean plane all have the form $f(z)=e^{i\theta}z+c$ or $f(z)=e^{i\theta}\overline{z}+c$ where α is a real number and c is a complex number. The converse of this conclusion also holds. Thus, every function of either of these forms is an isometry of the Euclidean plane.

CHAPTER 4: INVERSION OF THE EUCLIDEAN PLANE

This section describes a special type of mapping called an inversion. Inversions are conformal mappings, and unlike isometries they do not preserve distance in the Euclidean plane. Before we can define an inversion, we need the following propositions.

Proposition 4.1: *Let PO be a chord of a circle and let PT be any ray from P. Then, the line PT is tangent to the circle if and only if $\angle OPT$ is equal to the angle at the circumference subtended by the intercepted arc.*

Proof: Given a circle centered at C.



$(\Rightarrow)$ Let PO be a chord, let PT be a tangent line to the circle c, and let A be a point on the arc not enclosed by the angle $\angle TPO$ (see figure above). Let PR be a diameter. Then $\angle PAQ = \frac{1}{2}\angle PCQ = \angle PRQ$

Also $\angle TPR = \frac{\pi}{2}$, because PT is tangent to c. Therefore $\frac{\pi}{2} - \angle QPR = \angle TPQ$

Also note $\angle RQP$ is the angle in a semicircle and so $\angle RQP = \frac{\pi}{2}$.

Since the sum of the interior angles of a triangle is π, then

$$\angle PAQ = \angle PRQ = \frac{\pi}{2} - \angle QPR .$$

Thus $\angle PAQ = \angle TPQ$, so $\angle QPT$ is equal to the angle at the circumference subtended by the intercepted arc.

$(\Leftarrow)$ Let $\angle PAQ = \angle TPQ$ so that $\angle PRQ = \angle TPQ$. Thus it follows that

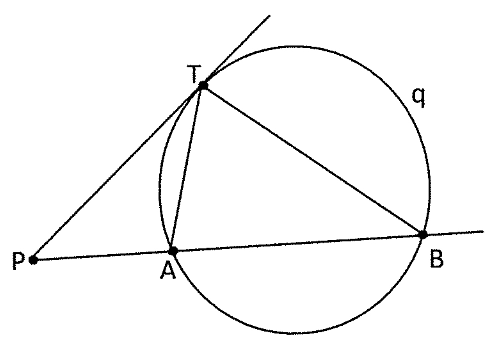
$$\angle TPR = \angle TPQ + \angle QPR = \angle PRQ + \angle QPR .$$

Since $\angle RQP = \frac{\pi}{2}$, and $\angle PRQ + \angle QPR = \pi - \angle RQP$ then $\angle TPR = \pi - \angle RQP = \frac{\pi}{2}$.

Thus PT is perpendicular to PR and so PT is tangent to the circle C . ■

Proposition 4.2: *Let P be a point outside a given circle q, let PT be a straight line, and let PAB be a secant with chord AB. Then, PT is tangent to q if and only if $PA \cdot PB = PT^2$.*

Proof: Let P be a point outside a given circle q, let PT be a straight line, and let PAB be a secant with cord AB.



($\Rightarrow$) Let PT be tangent to q. From the previous proposition, $\angle ATP = \angle PBT$. Since $\angle TPA$ is common to both ΔTPA and ΔBPT, then by AAA similarity $\Delta TPA \sim \Delta BPT$. Thus $\frac{PA}{PT} = \frac{PT}{PB}$ and so $PA \cdot PB = PT^2$.

($\Leftarrow$) Assume $PA \cdot PB = PT^2$. Then $\frac{PA}{PT} = \frac{PT}{PB}$. Since $\angle TPA$ is common to both ΔTPA and ΔBPT, then by SAS similarity $\Delta TPA \sim \Delta BPT$. Therefore $\angle ATP = \angle PBT$. Thus it follows from the previous proposition PT is tangent to circle q. ■

Now with the following definition we are able to define an inversion.

Definition 4.3: *Given a circle q with center C and radius k, the two points P and P' are symmetrical with respect to q, if*

i. C, P, *and* P' *are collinear, with* C *outside the segment* PP'

ii. $CP \cdot CP' = k^2$

Note that for a fixed circle q, the point P is symmetrical with the point P' if and only if the point P' is symmetrical with the point P. Also P is symmetrical with itself if and only if P lies on the circumference of q. Finally, it is clear that no point is symmetrical to C, and C is the only such point.

The inversion $I_{C,k}$ is undefined at C in the Euclidean plane. Consider the Euclidean plane as the plane of complex numbers and introduce an additional point, the point infinity. We will define the inversion of C to be the point infinity and the inversion of infinity to be C. The fixed points of $I_{C,k}$ are exactly those that lie on the circle centered at C. Thus an inversion is a map of this extended plane, called the *inversive plane.*

Definition 4.4: *Let C be a fixed point and k be a positive real number. The inversion is $I_{C,k}$ a function such that $I_{C,k}(P)=P'$ where P and P' are symmetrical with respect to the circle with center C and radius k.*

When the inversion transforms a circle p into a straight line m or vice versa, then the straight line that joins C to the center of the circle p is a line perpendicular to the given straight line m. When the inversion transforms a circle into a circle, their centers are collinear with C.

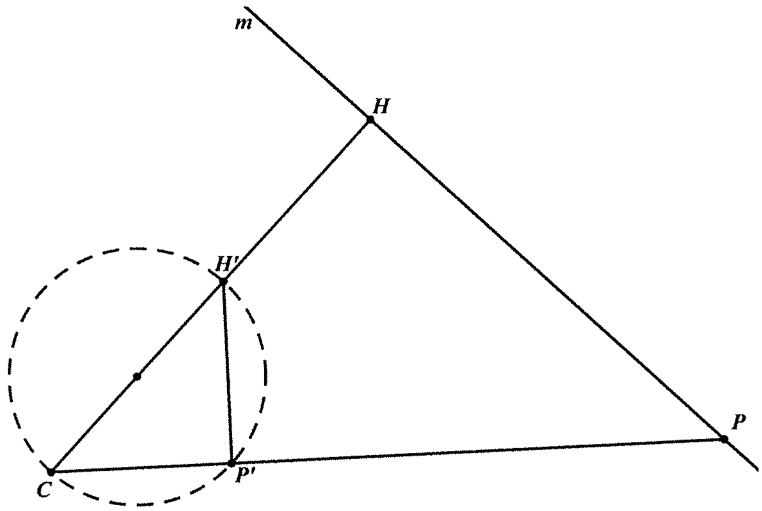
Theorem 4.5: *The inversion $I_{C,k}$ maps*

- *i.* *Straight line containing C onto themselves*
- *ii.* *Straight line not containing C onto circles through C*
- *iii.* *Circles through C onto straight line not containing C*
- *iv.* *Circles not through C onto circle not through C*

Proof:

Part *i*: This follows directly from the definition of an inversion.

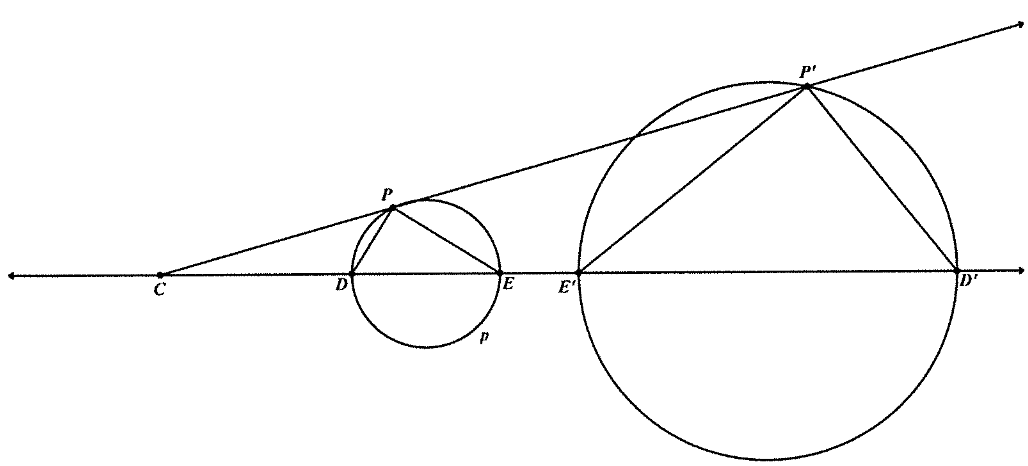
Part *ii*: Let m be a straight line not containing C, and let H be a point of m such that CH is perpendicular to m. Let P be any point on m. Let $H'=I_{C,k}(H)$ and $P'=I_{C,k}(P)$. Since $CH\cdot CH'=k^2$ and $CP\cdot CP'=k^2$, then $CH\cdot CH'=CP\cdot CP'$. Thus it follows that $\frac{CH'}{CP}=\frac{CP'}{CH}$. Also, since $\angle PCH$ is common to both $\triangle CHP$ and $\triangle CP'H'$, by SAS similarity $\triangle CHP \sim \triangle CP'H'$. Therefore $\angle CP'H'=\angle CHP=\frac{\pi}{2}$.



Since the position of H' is independent of those of P and P', it follows that the locus of P' is the circle that has CH' as its diameter.

Part *iii*: Since the inversion $I_{C,k}$ is an involution, then this is the analog from the proof of part *ii.* Instead of starting with a line not through C and mapping it to a circle through the C, we are starting with a circle through C and mapping it to a line not through C. Thus, we are done.

Part *iv*: Let p be a circle not containing C, and let P be an arbitrary point on P. Let DE be a diameter of P whose extension contains C, and let $D' = I_{C,k}(D)$, $E' = I_{C,k}(E)$, and $P' = I_{C,k}(P)$.



Since

$$CP \cdot CP' = k^2,\ CE \cdot CE' = k^2, \text{ and } CD \cdot CD' = k^2,$$

$CP \cdot CP' = CD \cdot CD' = CE \cdot CE'$. Thus it follows that $\frac{CD}{CP'} = \frac{CP}{CD'}$ and $\frac{CE}{CP'} = \frac{CP}{CE'}$. Since $\angle DCP$ is common to $\triangle DCP$, $\triangle D'CP'$, $\triangle ECP$, and $\triangle P'CE'$, then $\triangle DCP \sim \triangle D'CP'$ and $\triangle ECP \sim \triangle P'CE'$.

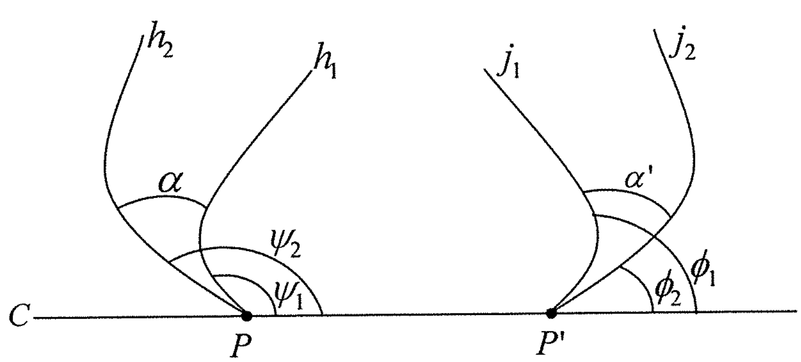
So, $\angle CDP = \angle CP'D'$ and $\angle CEP = \angle CP'E'$.

Note that $\angle E'P'D' = \angle CP'D' - \angle CP'E'$ and $\angle E'P'D' = \angle CDP - \angle CEP$. Since the exterior angle of a triangle is equal to the sum to the two interior angles, then $\angle CDP - \angle CEP = \angle DPE$. Thus $\angle E'P'D' = \angle DPE$. Also note that an angle subtended by a diameter at the circumference is $\frac{\pi}{2}$. Therefore $\angle DPE = \frac{\pi}{2}$ and so $\angle E'P'D' = \frac{\pi}{2}$. Since the positions of D' and E' are independent of those of P and P', then the points P', D', and E' form a circle with $D'E'$ as the diameter. ■

Recall that a conformal transformation of the plane is a mapping in the Euclidean plane that preserves the magnitude of the angles but not necessary the sign of the angle.

Theorem 4.6: *Inversions are conformal transformations of the plane.*

Proof:



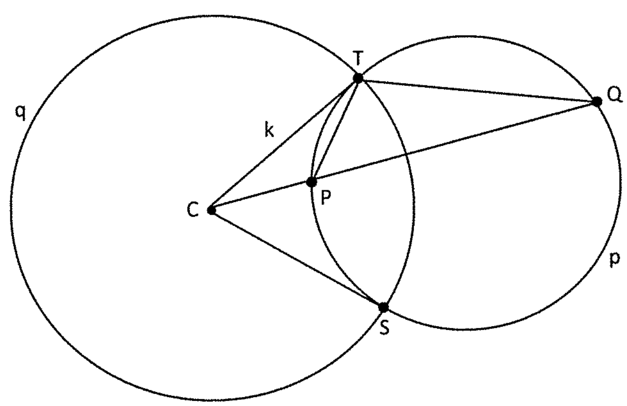
Let $I_{C,k}$ be an inversion. Place a polar coordinate system with its origin at C and its initial ray through the point P. Let h denote the curve $r=f(\theta)$, $\theta_o \le \theta \le \theta_1$. The inversion $I_{C,k}$ maps h to h' given by $r = F(\theta) = \frac{k^2}{f(\theta)}$, $\theta_o \le \theta \le \theta_1$. Suppose now that the angle α at P has sides h_i given by the equations $r=f_i(\theta)$ for $i=1, 2$ respectively. Suppose that the given inversion $I_{C,k}$ maps P, h_1, and h_2 onto P', j_1, and j_2 respectively. Then the image curves j_1 and j_2 have the equations $r = F_i(\theta) = \frac{k^2}{f_i(\theta)}$, $i=1,2$.

Note for any point P on any curve $r=f(\theta)$, the angle from the radius vector CP to the tangent line at P is given by $\tan(\psi) = \frac{r}{r'}$.

Therefore $\tan(\phi_1) = \frac{F_1}{F_1'} = \frac{k^2/f_1}{-k^2 f_1'/f_1^2} = -\frac{f_1}{f_1'} = -\tan(\psi_1)$. Hence $\phi_1 = \pi - \psi_1$ and similarly $\phi_2 = \pi - \psi_2$. Thus $\alpha' = \phi_1 - \phi_2 = (\pi - \psi_1) - (\pi - \psi_2) = \psi_2 - \psi_1 = \alpha$. ■

Proposition 4.7: *Let q be a circle with center C and radius k, and let p be any other circle. Then, the inversion $I_{C,k}$ fixes the circle p if and only if the circles p and q are orthogonal.*

Proof:



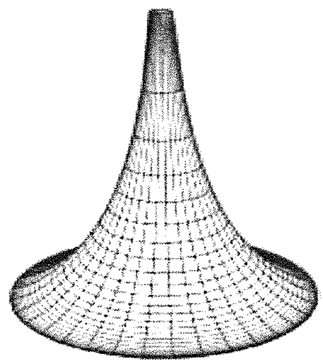
$(\Rightarrow)$ Let p be a circle orthogonal to q, and suppose they intersect at a point T. Let P be an arbitrary point of the circle p, and let Q be the other point where the secant line CP intersects the circle p. Therefore, by Proposition 4.2 $CP \cdot CQ = CT^2 = k^2$, and hence $I_{C,k}(P) = Q$. Thus the inversion $I_{C,k}$ maps the circle p onto itself.

$(\Leftarrow)$ Assume the circle p is fixed. Since $I_{C,k}$ interchanges points inside and outside of q, then the circle p intersect the circle q in two points, call these points T and S. Since $I_{C,k}$ fixes p and the straight lines CS and CT, then $I_{C,k}$ fixes the points T and S. Let P be any other point on p and let Q be the intersection of the secant line CP with the circle p. Hence, by Proposition 4.2 $CP \cdot CQ = CT^2 = k^2$, and thus it follows that $\frac{CP}{CT} = \frac{CT}{CQ}$. Since $\angle TCQ$ is common to both $\triangle CPT$ and $\triangle CTQ$, then $\triangle CTP \sim \triangle CTQ$. Thus $\angle CTP = \angle CQT$. So by the Proposition 4.1 CT is tangent to the circle p. Also note that the tangent to q at point T is perpendicular to CT. Since the tangents of p and q through T are perpendicular to each other, it follows that the circles p and q are also orthogonal to each other. ■

CHAPTER 5: THE PSEUDOSPHERE

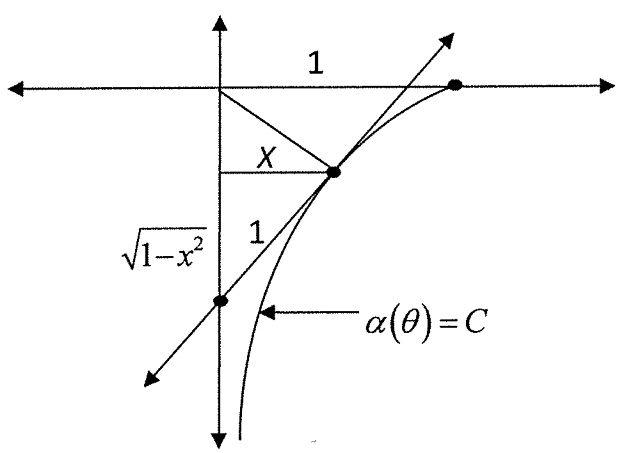
Beltrami called surfaces with constant negative curvature pseudospherical. In 1839 Minding proved that two surfaces with constant Gaussian curvatures are locally isometric if and only if their curvatures are equal. According to this result, all pseudospherical surfaces that have the same negative curvature value possess the same intrinsic geometry.

The simplest pseudospherical surface is the *pseudosphere.* Newton's definition of the pseudosphere states that the segment of the tangent from the point of contact to the *y*-axis has constant length R.



To construct the pseudosphere, first start with a tractrix. A tractrix is often described as the curve followed by a weight being dragged on the end of a fixed straight length, and the other end moves along a fixed straight line. Thus, the tractrix is a curve in which the segment of the tangent from the point of contact to the *y*-axis has constant length.

A tractrix can be parametrized as $\alpha(\theta)=\left(\sin(\theta),\log\left(\tan\left(\frac{\theta}{2}\right)\right)+\cos(\theta)\right)$ in the *xz* plane.



If we rotate $\alpha(\theta)$ about the line $z=0$ then we get the pseudosphere with $R=1$. Thus $S=\left(\sin(\theta)\cos(v),\sin(\theta)\sin(v),\log\left(\tan\left(\frac{\theta}{2}\right)\right)+\cos(\theta)\right)$ describes the pseudosphere.

So let $X(\theta,V)=\left(\sin\theta\cos V,\sin\theta\sin V,\log\left(\tan\left(\frac{\theta}{2}\right)\right)+\cos\theta\right)$, then it follows that

$$X_\theta=(\cos\theta\cos V,\cos\theta\sin V,\cot\theta\cos\theta)$$

$$X_V=(-\sin\theta\sin V,\sin\theta\cos V,0)$$

$$X_{\theta\theta}=(-\sin\theta\cos V,-\sin\theta\sin V,-\cos\theta-\cot\theta\csc\theta)$$

$$X_{VV}=(-\sin\theta\cos V,-\sin\theta\sin V,0)$$

$$X_{\theta V}=X_{V\theta}=(-\cos\theta\sin V,\cos\theta\cos V,0).$$

Note that $X_\theta\wedge X_V=(-\cos V\cos^2\theta,-\sin V\cos^2\theta,\cos\theta\sin\theta)$ and thus

$$|X_\theta\wedge X_V|=\sqrt{\cos^2 V\cos^4\theta+\sin^2 V\cos^4\theta+\cos^2\theta\sin^2\theta}=\cos\theta.$$

So then $N=\dfrac{X_\theta\wedge X_V}{|X_\theta\wedge X_V|}=\left(\dfrac{-\cos V\cos^2\theta}{\cos\theta},\dfrac{-\sin V\cos^2\theta}{\cos\theta},\dfrac{\cos\theta\sin\theta}{\cos\theta}\right)$. Therefore

$N=(-\cos V\cos\theta,-\sin V\cos\theta,\sin\theta)$.

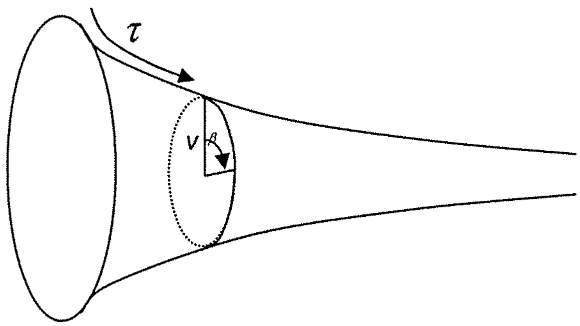
Then *e*, *f*, *g*, *E*, *F*, and *G* are computed as follows:

$$e=\langle N,X_{\theta\theta}\rangle=-\cot\theta,\ f=\langle N,X_{\theta V}\rangle=0,\text{ and } g=\langle N,X_{VV}\rangle=\sin\theta\cos\theta$$

$$E=\langle X_\theta,X_\theta\rangle=\cos^2\theta\csc^2\theta,\ F=\langle X_\theta,X_V\rangle=0,\text{ and } G=\langle X_V,X_V\rangle=\sin^2\theta.$$

Finally, $K = \frac{eg - f^2}{EG - F^2} = \frac{-\cot\theta\cos\theta\sin\theta - 0}{\cos^2\theta\csc^2\theta\sin^2\theta - 0} = -1$. Therefore the pseudosphere has constant curvature -1.

Next consider the following diagram.



The following parametric equations for the tractrix can be obtained: $u = \sigma - \tanh\sigma$ and $v = \operatorname{sech}\sigma$. Then we can use u and v to find the arc length τ along the tractrix to be $\tau = \int_0^\sigma \sqrt{du^2 + dv^2} = \log(\cosh\sigma)$. Thus $\cosh\sigma = e^\tau$ and hence $v = e^{-\tau}$. So now use τ and the angle β as coordinates on the pseudosphere. Therefore the length subtended by the angle dx on a circular cross section is $vdx = e^{-\tau}dx$ and hence the infinitesimal distance ds between the points (x,τ) and $(x + dx, \tau + d\tau)$ is $ds^2 = e^{-2\tau}dx^2 + d\tau^2$. Lastly we will introduce the variable $y = e^\tau$ which implies that $dy = e^\tau d\tau$.

Once we make the substitution we get:

$$ds^2 = e^{-2\tau}dx^2 + d\tau^2 = e^{-2\tau}dx^2 + e^{-2\tau}dy^2 = e^{-2\tau}\left(dx^2 + dy^2\right) = \frac{dx^2 + dy^2}{y^2}.$$

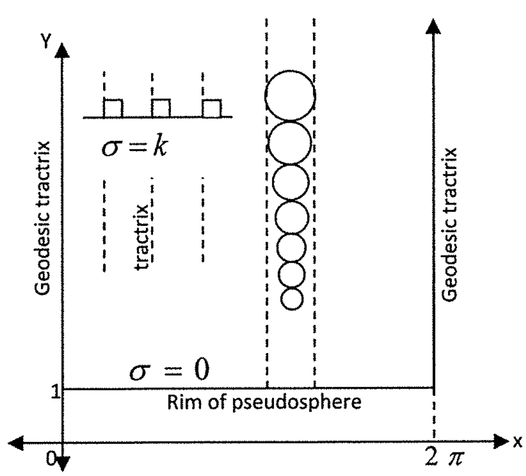
Therefore the pseudosphere is locally isometric to the *xy*-plane if the distance is defined to be $ds^2 = \frac{dx^2 + dy^2}{y^2}$.

Now we can describe a conformal map of the pseudosphere. First the tractrix generators $x = c$, where c is a constant, are orthogonal to the circular cross sections $\sigma = k$, where k is a constant. So the image of $\sigma = k$ is represented by a horizontal line $\sigma = k$.

Secondly consider the arc of the circle $\sigma = k$ connecting the points (x,σ) and $(x+dx,\sigma)$. The separation on the pseudosphere is xdx. In the image of the map these two points have the same height and are separated by distance dx. Thus in going from the pseudosphere to the image of the map this line segment is shrunk by a factor of x. However, since the map is conformal, an infinitesimal line segment emanating from (x,σ) in any direction must be multiplied by the same factor $\frac{1}{x} = e^{\sigma}$. Therefore the metric is $\widehat{ds} = Xds$.

Thirdly, consider a disc on the upper part of the pseudosphere. With a diameter ε. In the image of the map it will be represented by another disc, whose diameter is $\frac{\varepsilon}{X}$. Now suppose we repeatedly translate the original disc toward the pseudosphere rim, moving it a distance of ε each time. As the disc moves down the pseudosphere, it reaches the axis and its angular width diminishes. Thus the image disc in the map appears to gradually shrink as it moves downward.

Thus $y = e^{\sigma} = \left(\frac{1}{x}\right)$ is the y-coordinate corresponding to the point (x,σ) on the pseudosphere. Therefore the entire pseudosphere is represented by the shaded area above the line $y = 1$ and represents the rim of the pseudosphere.



The metric associated with the map is $\widehat{ds} = \frac{ds}{y} = \frac{\sqrt{dx^2 + dy^2}}{y}$.

In 1868, Beltrami discovered that hyperbolic geometry could be given a concrete interpretation, and he also discovered that figures drawn on the pseudosphere obey the rules of hyperbolic geometry.

One problem with the pseudosphere is that it has a boundary, the rim. Therefore it is not a complete surface of the hyperbolic plane. We want a hyperbolic plane to not have a boundary. The abstract hyperbolic geometry that was discovered by Gauss, Bolyai, and Lobachevsky is understood to take place in the hyperbolic plane. This hyperbolic plane is meant to be exactly like the Euclidean plane except that lines within the hyperbolic plane do not conform to the fifth postulate and obey the following axiom: given a line L and a point p not on L , there are at least two lines through p that do not meet L .

The pseudosphere is topologically a cylinder rather than a plane. For example, a closed loop in the hyperbolic plane can be shrunk to a single point, whereas this doesn't happen for a loop on the pseudosphere that wraps around the axis. Secondly, in the hyperbolic plane a line-segment can be extended indefinitely in either direction, but on the pseudosphere vertical lines cannot be extended indefinitely in both direction. They will terminate when they reach the rim. Since the hyperbolic plane differs from the Euclidean plane in two ways, the pseudosphere will not model the entire hyperbolic plane.

Beltrami was able to solve the first of these problems. Imagine a sheet wrapping around the pseudosphere infinitely many times. Then unwrap the sheet and cover the entire region above the line $y = 1$, stretching as you go. As a result, a particle traveling round and round a circle in the map would correspond to a particle traveling along a horizontal line $\sigma = k$ on the pseudosphere.

Now the conformal map we developed solves the second problem. Imagine yourself as a tiny two-dimensional being living in the hyperbolic plane, walking down a line $x = c$

on the pseudosphere. Your walk will eventually be interrupted by the pseudosphere rim at some point $\hat{p}$, which corresponds to a point P on the line $y = 1$. But in the map this point p is just like any other, so there is nothing preventing you from continuing your walk down toward the line $y = 0$. You will never actually reach the line $y = 0$ because it is infinitely far from P. This is a consequence of the defined metric.

We will restrict the Euclidean plane to the upper half-plane and refer to it with the hyperbolic metric as the hyperbolic plane. This is also known as the Poincare upper half-plane. There are several other different models of the hyperbolic plane, which include Poincare's disk model and the Beltrami-Klein model.

CHAPTER 6: THE POINCARE HALF-PLANE

The Poincare half-plane model is denoted by $\mathbb{H}^2$ and is the set of points $\mathbb{H}^2 = \{(u,v) \in \mathbb{R}^2 \mid v > 0\}$ with the hyperbolic metric, $\frac{\sqrt{dx^2 + dy^2}}{y}$.

First let's find the curvature in the upper half plane. Note that $E = \frac{1}{v^2}$, $F = 0$, and $G = \frac{1}{v^2}$ in the upper half plane model the curvature can be found by the equation $K = \frac{-1}{2\sqrt{EG}}\left\{\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right\}$.

Also note that $E_v = \frac{-2}{v^3}$ and $G_u = 0$. Thus,

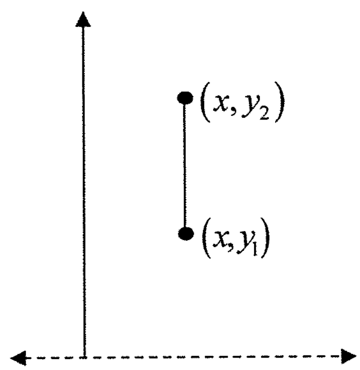
$$K = \frac{-1}{2\sqrt{\frac{1}{v^4}}}\left\{\left(\frac{\frac{-2}{v^3}}{\sqrt{\frac{1}{v^4}}}\right)_v + 0\right\} = \frac{-v^2}{2}\left\{\left(\frac{-2}{v}\right)_v\right\} = \frac{-v^2}{2}\cdot\frac{2}{v^2} = -1.$$

Therefore the upper half plane has constant curvature -1, just like the pseudosphere. Since the metric on $\mathbb{H}^2$ was developed from the metric on the pseudosphere, then we knew this should be the case.

Now let's distinguish the hyperbolic plane from the Euclidean plane. This difference is the way distance is measured: $\text{hyperbolic length} = \frac{\text{Euclidean length}}{y}$.

Therefore $ds = \dfrac{\sqrt{dx^2 + dy^2}}{y}$. We use calculus to compute the hyperbolic length for any curve. We define the hyperbolic length of an arbitrary curve γ as $\displaystyle\int_{\gamma} \frac{\sqrt{dx^2 + dy^2}}{y}$.

First we will find the length of a vertical line in the hyperbolic plane.



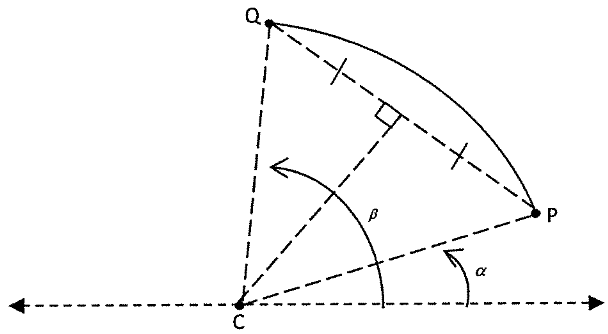
Let $x = f(y)$ and thus $dx = f'(y)dy$. Therefore the length of the segment is $\displaystyle\int_{y_1}^{y_2} \frac{\sqrt{dx^2 + dy^2}}{y} = \int_{y_1}^{y_2} \frac{\sqrt{(f'(y)dy)^2 + dy^2}}{y} = \int_{y_1}^{y_2} \frac{\sqrt{(f'(y))^2 + 1}}{y} dy$. Since the line segment is vertical, we have $f'(y) = 0$. Thus, $\displaystyle\int_{y_1}^{y_2} \frac{1}{y} dy = \ln y \,|_{y_1}^{y_2} = \ln\left(\frac{y_2}{y_1}\right)$.

Fix y_1 and then $\displaystyle\lim_{y_2 \to \infty} \ln\left(\frac{y_2}{y_1}\right) = \infty$. This shows that there is no upper bound in the upper-half plane model of the hyperbolic plane.

Now fix y_2 and then $\displaystyle\lim_{y_1 \to 0^+} \ln\left(\frac{y_2}{y_1}\right) = \infty$. So the x-axis is also not a boundary in the upper-half plane model of the hyperbolic plane.

Proposition 6.1: *Let q be a circle with center $C(c,0)$ and Euclidean radius r . If P and Q are points of q such that the radii CP and CQ make angles α and β with $\alpha<\beta$ respectively, with the positive x-axis, then* $L_h(PQ)=\ln\left(\dfrac{\csc\beta-\cot\beta}{\csc\alpha-\cot\alpha}\right)$.

Proof:



If t is the angle from the positive x-axis to the radius through an arbitrary point (x,y) on q, then $x=c+r\cos(t)$ and $y=r\sin(t)$. Thus $dx=-r\sin(t)dt$ and $dy=r\cos(t)dt$. Then it follows that the hyperbolic length of the arc PQ is

$$\int_\alpha^\beta \frac{\sqrt{(-r\sin(t)dt)^2+(r\cos(t)dt)^2}}{r\sin(t)}=\int_\alpha^\beta \frac{rdt}{r\sin(t)}=\int_\alpha^\beta \csc(t)dt=\ln\left(\frac{\csc(\beta)-\cot(\beta)}{\csc(\alpha)-\cot(\alpha)}\right).\ \blacksquare$$

Fix β , then consider $\lim_{\alpha\to0^+}\ln\dfrac{\csc\beta-\cot\beta}{\csc\alpha-\cot\alpha}$. The numerator, $\csc\beta-\cot\beta$ is constant. Now consider $\lim_{\alpha\to0^+}\csc\alpha-\cot\alpha=\lim_{\alpha\to0^+}\dfrac{1-\cos\alpha}{\sin\alpha}=\lim_{\alpha\to0^+}\dfrac{\sin\alpha}{\cos\alpha}=0$. Thus the denominator, $\csc\alpha-\cot\alpha$ becomes extremely small, and so $\lim_{\alpha\to0^+}\ln\dfrac{\csc\beta-\cot\beta}{\csc\alpha-\cot\alpha}=\infty$. Therefore the point P denoted in the previous theorem never touches the x-axis.

Now fix α, and consider, $\lim_{\beta\to\pi}\left(\ln\frac{\csc(\beta)-\cot(\beta)}{\csc(\alpha)-\cot(\alpha)}\right)$. The denominator, $\csc(\alpha)-\cot(\alpha)$ is constant. Then $\lim_{\beta\to\pi}(\csc(\beta)-\cot(\beta))=\lim_{\alpha\to 0^+}\left(\frac{1-\cos(\alpha)}{\sin(\alpha)}\right)=\infty$. Thus the numerator $\csc(\beta)-\cot(\beta)$ becomes extremely large and so $\lim_{\beta\to\pi}\left(\ln\frac{\csc(\beta)-\cot(\beta)}{\csc(\alpha)-\cot(\alpha)}\right)=\infty$. This implies that the point Q from the previous theorem will never touch the x-axis either.

Note that there are only these two cases. We can have the two points above each other to form a vertical line segment or the two points are not above each other and hence the line connecting them is the arc of a circle with the center on the x-axis. We will call geodesics that are vertical lines, straight geodesics and we will call geodesics that are part of an arc of a circle with the center on the x-axis, bowed geodesics.

Theorem 6.2: *The following Euclidean transformations of the hyperbolic plane preserve both hyperbolic lengths and measures of angles.*

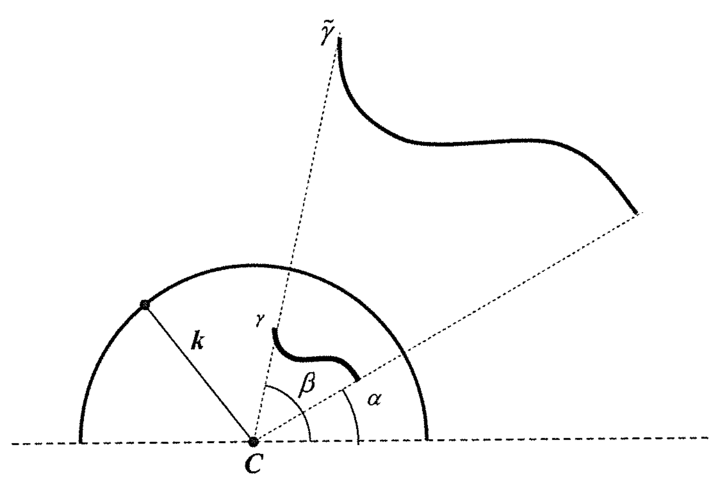
i. *Inversions $I_{C,k}$, where C is on the x-axis*

ii. *Reflections ρ_m, where m is perpendicular to the x-axis*

iii. *Translations τ_{AB}, where AB is parallel to the x-axis*

Proof:

Part *i*:



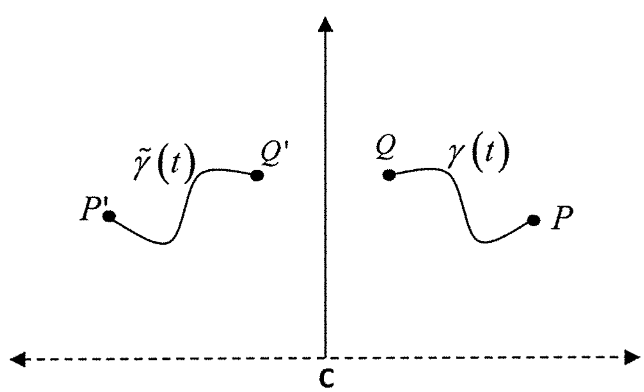
Let $I_{C,k}$ be the given inversion. Let γ be the curve $r=f(\theta)$ and let $\tilde{\gamma}$ be the curve $r=F(\theta)$, the image of γ after the inversion $I_{C,k}$. Then $f(\theta)\cdot F(\theta)=k^2$ and thus $F(\theta)=\frac{k^2}{f(\theta)}$, which implies that $F'(\theta)=\frac{-k^2 f'(\theta)}{(f(\theta))^2}$. The hyperbolic length of $\tilde{\gamma}$ is

$$\int_\alpha^\beta \frac{\sqrt{[F'(\theta)]^2+[F(\theta)]^2}}{F(\theta)\sin\theta}d\theta = \int_\alpha^\beta \frac{\sqrt{\left[\frac{-k^2 f'(\theta)}{[f(\theta)]^2}\right]^2+\left[\frac{k^2}{f(\theta)}\right]^2}}{\left[\frac{k^2}{f(\theta)}\right]\sin\theta}d\theta$$

$=\int_\alpha^\beta \frac{\sqrt{[f'(\theta)]^2+[f(\theta)]^2}}{f(\theta)\sin\theta}d\theta$,which is the hyperbolic length of γ.

Therefore Euclidean inversions preserve hyperbolic length.

Part *ii*: Let the vertical line m have the equation $x=c$.



It is clear that two points (x_1,y_1) and (x_2,y_2) are symmetrical with respect to this line if and only if $c=\frac{x_1+x_2}{2}$ and $y_1=y_2$. Let γ be a curve parameterized as $(u(t),v(t))$, $a<t<b$ and $\tilde{\gamma}=\rho_m(\gamma)$ has the parameterization $(2c-u(t),v(t))$, $a<t<b$.

Thus it follows that along γ, $dx = u'(t)dt$ and $dy = v'(t)dt$, and along $\tilde{\gamma}$, $dx = -u'(t)dt$ and $dy = v'(t)dt$.

Therefore the length of γ is $\int_P^Q \frac{\sqrt{dx^2+dy^2}}{y} = \int_a^b \frac{\sqrt{u'(t)^2+v'(t)^2}}{v(t)}dt$ and the length of $\tilde{\gamma}$ is $\int_{P'}^{Q'} \frac{\sqrt{dx^2+dy^2}}{y} = \int_a^b \frac{\sqrt{[-u'(t)]^2+v'(t)^2}}{v(t)}dt$.

Since $\int_a^b \frac{\sqrt{[u'(t)]^2+v'(t)^2}}{v(t)}dt = \int_a^b \frac{\sqrt{[-u'(t)]^2+v'(t)^2}}{v(t)}dt$, the length of γ is equal to the length of $\tilde{\gamma}$; hence Euclidean reflection over a vertical line preserves hyperbolic length.

Part *iii*: Let τ be the translation such that $\tau(x,y) = (x+h,y)$ for some fixed number h. If γ is any curve parameterized at $(u(t),v(t))$, $a<t<b$, then $\tau(\gamma) = \tilde{\gamma}$ has the parameterization $(u(t)+h,v(t))$, $a<t<b$. Thus it follows along both γ and $\tilde{\gamma}$ that $dx = u'(t)dt$ and $dy = v'(t)dt$. Therefore the length of γ is $\int_a^b \frac{\sqrt{u'(t)^2+v'(t)^2}}{v(t)}dt$ which is the length of $\tilde{\gamma}$ as well. ■

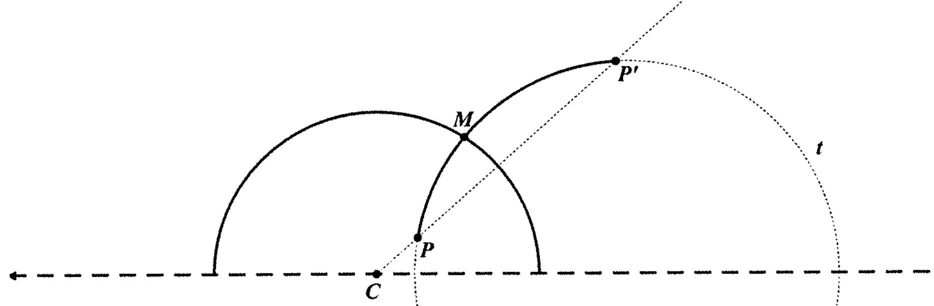
Theorem 6.3: *A hyperbolic reflection is either*

i. A Euclidean inversion, $I_{C,k}$*, where* C *is on the x-axis.*

ii. A Euclidean reflection over a vertical line.

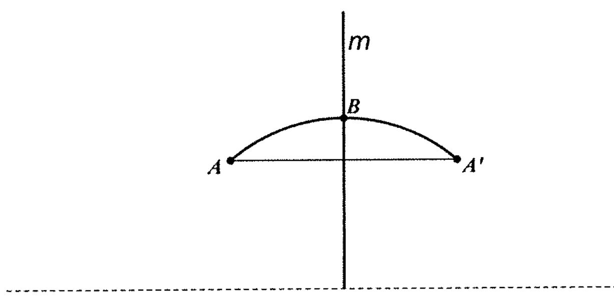
Proof:

Part *i:*



In the previous theorem we showed that for $I_{c,k}$, where C is on the x-axis , the hyperbolic length of finite curves is preserved. We need to show that $PM = MP'$. Note $I_{c,k}$ maps circles to circles, and so $I_{c,k}$ maps circle t to a circle. In fact since P is mapped to P', P' is mapped to P and M is a fixed point, then circle t is mapped to itself under $I_{c,k}$. So arc PM maps to arc MP'. Since $I_{c,k}$ fixes circle t, then the angle of the intersection is $\frac{\pi}{2}$, by Proposition 4.7. We know angles in the hyperbolic plane have the same measure as their measure in the Euclidean plane, so the angle at point M is $\frac{\pi}{2}$. Therefore q is a perpendicular bisector of PP', which implies that $I_{c,k}$ is a hyperbolic reflection.

Part *ii:*



From Theorem 6.2, we know that the hyperbolic length of the curve AB is the same length as the curve $A'B$. Note that the hyperbolic line is the arc of a circle centered on the *x*-axis, and since $AB = A'B$ then line m goes through the center of the circle containing A, B, and A'. Therefore the angle at B is $\frac{\pi}{2}$. Hence line m is the perpendicular bisector of the hyperbolic line AA'. Thus, a Euclidean reflection over a vertical line is a hyperbolic reflection. ■

Theorem 6.4: *Each* $\mathbb{H}^2$- *isometry is the composition of zero, one, two or three hyperbolic reflections.*

Proof: Since $\mathbb{H}^2$ geometry contains absolute geometry, then the proof is the same as the proof of Theorem 2.7.

Corollary 6.5: *The set of all* $\mathbb{H}^2$*- isometries form a group.*

Proof: The proof follows the same idea as in the Euclidean case because each reflection is its own inverse and function composition is associative. Also the identity is a hyperbolic isometry and the composition of isometries is an isometry. ∎

CHAPTER 7: MÖBIUS TRANSFORMATIONS

We denote $\mathrm{M} = \left\{ T \mid \mathbb{C} \to \mathbb{C} \text{ where } Tz = \dfrac{\alpha z + \beta}{\gamma z + \delta} \text{ with } \alpha\delta - \beta\gamma \neq 0 \right\}$ to be the set of all Möius transformations.

If $\alpha\delta - \beta\gamma = 0$, then $\alpha\delta = \beta\gamma$. Therefore

$$Tz = \frac{\alpha z + \beta}{\gamma z + \delta} \cdot \frac{\gamma}{\gamma} = \frac{(\alpha\gamma) z + \beta\gamma}{\gamma^2 z + \gamma\delta} = \frac{(\alpha\gamma) z + \alpha\delta}{\gamma^2 z + \gamma\delta} = \frac{\alpha(\gamma z + \delta)}{\gamma(\gamma z + \delta)} = \frac{\alpha}{\gamma}$$, which is a constant.

So the complex plane would be mapped to a single point and thus this is the trivial case so it is not interesting. Consider the special cases below.

The first type is called a *dilation* about 0 with $r > 0$. The dilation is called a stretch if $r > 1$ of shrink if $0 < r < 1$. If we let $\beta = 0 = \gamma$, then a dilation is $Tz = \left(\dfrac{\alpha}{\delta}\right) z = \alpha' z$.

The second type is called a *translation* of the complex plane, which is described as the shifting of z in some direction. So let $\gamma = 0,\ \alpha = 1 = \delta$ and thus a translation is denoted as $tz = z + \beta$.

The third type is called *reciprocation.* If we let $\alpha = \delta = 0$ and $\beta = \gamma = 1$, then we describe a reciprocation by $Tz = \dfrac{1}{z}$.

Therefore infinity maps to 0, 0 maps to infinity, the unit circle is mapped to itself, the inside of the unit circle is mapped to the outside of the unit circle, and the outside of the unit circle is mapped to the inside of the unit circle.

Finally a *reflection* about the *y*-axis is $Tz = -\overline{z}$. By taking the negative of z, it is a reflection about the origin and then the conjugate of z reflects it about the *x*-axis.

Theorem 7.1: *Every Möbius Transformation is a composition of at most dilations, translations, and reciprocations.*

Proof: Case 1: $\gamma=0$ and $Tz=\dfrac{\alpha z+\beta}{\gamma z+\delta}, \alpha\delta-\beta\gamma\neq 0$ Then $\alpha\neq 0$ and $\delta\neq 0$. So $Tz=\left(\dfrac{\alpha}{\delta}\right)z+\left(\dfrac{\beta}{\delta}\right)$. Let $T_1z=\left(\dfrac{\alpha}{\delta}\right)z\in \mathrm{M}$ and let $T_2z=z+\left(\dfrac{\beta}{\delta}\right)\in \mathrm{M}$, therefore $Tz=(T_2\circ T_1)z$.

Case 2: $\gamma\neq 0$. Therefore $Tz=\dfrac{\alpha z+\beta}{\gamma z+\delta}=\dfrac{\alpha}{\gamma}+\dfrac{\beta-\dfrac{\alpha\delta}{\gamma}}{\gamma z+\delta}$.

Let

$$T_1z=\gamma z\ \text{(dilation)}$$

$$T_2z=z+\delta\ \text{(translation)}$$

$$T_3z=\frac{1}{z}\ \text{(reciprocation)}$$

$$T_4z=\left(\beta-\frac{\alpha\delta}{\gamma}\right)z\ \text{(dilation)}$$

$$T_5z=z+\frac{\alpha}{\gamma}\ \text{(translation)}$$

So $Tz=(T_5\circ T_4\circ T_3\circ T_2\circ T_1)z$. Then $T_1z=\gamma z$ and so $T_2(T_1z)=\gamma z+\delta$. Then $T_3(T_2(T_1z))=\dfrac{1}{\gamma z+\delta}$. Then $T_4(T_3(T_2(T_1)))=\dfrac{\left(\beta-\dfrac{\alpha\delta}{\gamma}\right)}{\gamma z+\delta}$ and finally

$$T_5(T_4(T_3(T_2(T_1))))=\frac{\beta-\dfrac{\alpha\delta}{\gamma}}{\gamma z+\delta}+\frac{\alpha}{\gamma}=\frac{\gamma\left(\beta-\dfrac{\alpha\delta}{\gamma}\right)+\alpha(\gamma z+\delta)}{\gamma(\gamma z+\delta)}$$

$$=\frac{\gamma\beta-\alpha\delta+\alpha\gamma z+\alpha\delta}{\gamma(\gamma z+\delta)}=\frac{\gamma(\alpha z+\beta)}{\gamma(\gamma z+\delta)}$$

Therefore $T_5\left(T_4\left(T_3\left(T_2\left(T_1\right)\right)\right)\right)=\dfrac{\alpha z+\beta}{\gamma z+\delta}$.■

Now consider the domain and range of Möbius transformations.

Since we cannot divide by zero $\gamma z+\delta \neq 0$ and $z \neq \dfrac{-\delta}{\gamma}$; thus the domain is $\mathbb{C}\setminus\left\{-\dfrac{\delta}{\gamma}\right\}$.

If we divide by z we get $Tz = \dfrac{\alpha + \dfrac{\beta}{z}}{\gamma + \dfrac{\delta}{z}}$ and so the range is $\mathbb{C}\setminus\left\{\dfrac{\alpha}{\gamma}\right\}$.

So $T:\mathbb{C}\setminus\left\{\dfrac{-\delta}{\gamma}\right\}\to\mathbb{C}\setminus\left\{\dfrac{\alpha}{\gamma}\right\}$. Now extend $\mathbb{C}$ to the extended complex plane, $\hat{\mathbb{C}}$, and hence

$$\hat{T}z=\begin{cases} Tz \text{ if } z\in\mathbb{C}\setminus\left\{-\dfrac{\delta}{\gamma}\right\} \\ \dfrac{\alpha}{\gamma} \text{ if } z=\infty \\ \infty \text{ if } z=\dfrac{-\delta}{\gamma} \end{cases}$$

Theorem 7.2: *If* $Tz=\dfrac{\alpha z+\beta}{\gamma z+\delta},\alpha\delta-\beta\gamma\neq 0$ *, then* $\hat{T}:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ *is a bijection.*

Proof: There exists $z_1,z_2\in\mathbb{C}\setminus\left\{-\dfrac{\delta}{\gamma}\right\}$ with $Tz_1=Tz_2$. This implies $\dfrac{\alpha z_1+\beta}{\gamma z_1+\delta}=\dfrac{\alpha z_2+\beta}{\gamma z_2+\delta}$

which implies $\left(\alpha z_1+\beta\right)\left(\gamma z_2+\delta\right)=\left(\gamma z_1+\delta\right)\left(\alpha z_2+\beta\right)$ which leads to

$$\alpha\gamma z_1z_2+\alpha\delta z_1+\beta\gamma z_2+\beta\delta=\alpha\gamma z_1z_2+\alpha\delta z_2+\beta\gamma z_1+\beta\delta$$

$$\alpha\delta z_1-\beta\gamma z_1=\alpha\delta z_2-\beta\gamma z_2$$

$$\left(\alpha\delta-\beta\gamma\right)z_1=\left(\alpha\delta-\beta\gamma\right)z_2.$$

Since $\alpha\delta-\beta\gamma\neq 0$ then $z_1=z_2$. Thus T is one to one.

Let $w\in\mathbb{C}\setminus\left\{\frac{\alpha}{\gamma}\right\}$. If $z:Tz=w$ then $\frac{\alpha z+\beta}{\gamma z+\delta}=w$ which implies $\alpha z+\beta=\gamma zw+\delta w$ and so $\alpha z-\gamma zw=+\delta w-\beta$. Thus $(\alpha-\gamma w)z=\delta w-\beta$ and so $z=\frac{\delta w-\beta}{\alpha-\gamma w}\in\mathbb{C}\setminus\left\{\frac{-\delta}{\gamma}\right\}$. Since $\alpha\delta-(-\gamma)(-\beta)=\alpha\delta-\beta\gamma\neq 0$, then z is a Möbius transformation. Hence T is onto.

Therefore T is bijective. ∎

Corollary 7.3: *The compositional inverse of a Möbius transformation is a Möbius transformation.*

Lemma 7.4: *If $T,S\in\mathrm{M}$, then $T\circ S\in\mathrm{M}$*

Proof: Let $Tz=\frac{\alpha z+\beta}{\gamma z+\delta}, \alpha\delta-\beta\gamma\neq 0$ and let $Sz=\frac{az+b}{cz+d}, ad-bc\neq 0$.

Then

$$(T\circ S)z=T(Sz)=T\left(\frac{az+b}{cz+d}\right)=\frac{\alpha\left(\frac{az+b}{cz+d}\right)+\beta}{\gamma\left(\frac{az+b}{cz+d}\right)+\delta}\cdot\left(\frac{cz+d}{cz+d}\right)$$

$$=\frac{\alpha az+\alpha b+\beta cz+\beta d}{\gamma az+\gamma b+\delta cz+\delta d}=\frac{(\alpha a+\beta c)z+(\alpha b+\beta d)}{(\gamma a+\delta c)z+(\gamma b+\delta d)}$$

Thus it remains to check that $(\alpha a+\beta c)(\gamma b+\delta d)-(\alpha b+\beta d)(\gamma a+\delta c)\neq 0$.

$$(\alpha a+\beta c)(\gamma b+\delta d)-(\alpha b+\beta d)(\gamma a+\delta c)$$
$$=\alpha\gamma ab+\alpha\delta ad+\beta\gamma bc+\beta\delta cd-\alpha\delta ab-\alpha\delta bc-\beta\gamma ad-\beta\delta cd$$
$$=\alpha\delta ad+\beta\gamma bc-\alpha\delta bc-\beta\gamma ad$$
$$=\alpha\delta(ad-bc)-\beta\gamma(ad-bc)=(\alpha\delta-\beta\gamma)(ad-bc)\neq 0$$

Therefore $(T\circ S)z\in \mathrm{M}$. ■

Theorem 7.5: *The set of Möbius transformations with composition is a group on* $\hat{\mathbb{C}}$.

Proof : 1. Closure: by previous lemma

2. Associativity: composition of functions is associative

3. Identity: $Tz=z=\dfrac{1z+0}{0z+1}\in \mathrm{M}$

4. Inverse: yes, by the previous corollary .

So from this point we develop a natural question; what do translations, dilations, and reciprocations do to a circle or line?

First start with translations, $Tz=z+\beta$ where $\beta=x_o+iy_o$

Let $L=\{z=x+iy\mid y=mx+b\}$ which is the set of lines in the complex plane. Then

$$Tz=z+\beta=(x+iy)+(x_o+iy_o)=(x+x_o)+i(y+y_o)$$

And therefore, $m(x+x_o)+b=mx+mx_o+b=y+mx_o$, since $y=mx+b$. Therefore

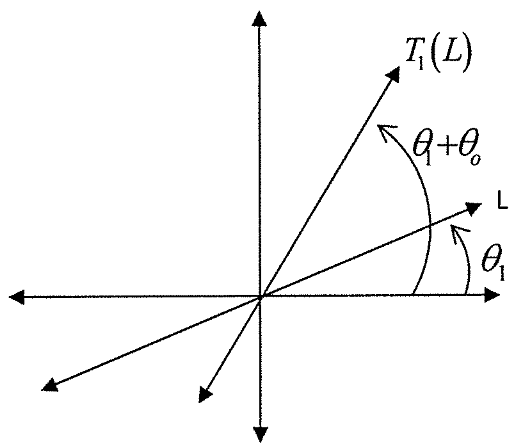
$$y+mx_o=y+y_o-y_o+mx_o=(y+y_o)-(-mx_o+y_o).$$

Thus $m\overbrace{(x+x_o)}^{newx}+\overbrace{(-mx_o+y_o)}^{newb}=\overbrace{y+y_o}^{newy}$ is a line with the same slope and different y-intercept.

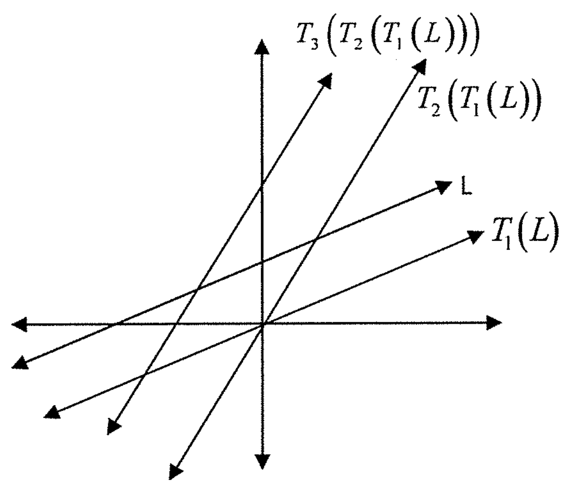
Therefore translations map lines to lines.

Let $C=\{z=z_1+re^{it},0\le t\le 2\pi\}$ and let $z\in C$ and hence $Tz=z+\beta=(z_1+\beta)+re^{it}$ which is a circle with radius *r*, and center $z_1+\beta$. Thus translations map circles to circles. Next, consider dilations, $Tz=\alpha z$ with $\alpha\ne 0$. Let $\alpha=r_o e^{i\theta_o}$.

Case 1. Let L be a line through $(0,0)$, so $L=\{re^{i\theta}\,|-\infty<r<\infty\}$. Let $z\in L$. Then $Tz=\alpha z=(r_o e^{i\theta_o})(re^{i\theta})=r_o re^{i(\theta_o+\theta)}$. Thus we get a line through $(0,0)$ and $-\infty<r<\infty$ at a different angle.



Case 2. Start with some L through b, then apply $T_1 z=z-b$ to L which is a translation of L to the origin. Then apply $T_2 z=\alpha z$ which is a line through the center at a different angle.



Finally apply $T_3 z = z + \alpha b$ which is another translation, and hence $(T_3 \circ T_2 \circ T_1) z = T_3(T_2(z-b)) = T_3(\alpha z - \alpha b) = \alpha z$. Therefore a line not through the origin is mapped to a line not through the origin. Thus a dilation maps lines to lines.

Let $C = \{z_1 + re^{i\theta} \mid 0 \le \theta \le 2\pi\}$ and let $z \in C$.

Then

$$Tz = T\left(z_1 + re^{i\theta}\right) = \alpha\left(z_1 + re^{i\theta}\right) = \alpha z_1 + \alpha re^{i\theta} = \alpha z_1 + r_o re^{i(\theta+\theta_o)}$$

where $0 \le \theta \le 2\pi$ so Tz is centered at αz_1 and has a radius rr_o. Therefore dilations map circles to circles.

Finally we take a look at reciprocations, $Tz = \frac{1}{z}$.

Note that the general equation of a circle or a line in $\mathbb{R}^2$ is $A(x^2 + y^2) + Bx + Cy + D = 0$. Since we do not want a point or the empty set, then we will require $B^2 + C^2 - 4AD > 0$ and $A, B, C, D \in \mathbb{R}$. If $A = 0$, then we get a line and if $A \ne 0$ then we get a circle. So in the complex plane the equation of a circle or line is $A(|z|^2) + \beta\left(\frac{z+\overline{z}}{2}\right) + C\left(\frac{z-\overline{z}}{2i}\right) + D = 0$. So $A|z|^2 + z\left(\frac{\beta}{2} + \frac{C}{2i}\right) + \overline{z}\left(\frac{\beta}{2} - \frac{C}{2i}\right) + D = 0$ and hence $Az\overline{z} + z\left(\frac{\beta - iC}{2}\right) + \overline{z}\left(\frac{\beta + iC}{2}\right) + D = 0$.

Let $a = A$, $d = D$, and $b = \frac{\beta + iC}{2}$. Therefore, the general equation of a circle or line is: $az\overline{z} + \overline{b}z + b\overline{z} + d = 0$ where $a, d \in \mathbb{R}$ and $b \in \mathbb{C}$.

So $Tz = \frac{1}{z}$ and let L be a circle or a line and let $z \in L$ which implies $az\overline{z} + \overline{b}z + b\overline{z} + d = 0$. Let $w = Tz$ so $w = \frac{1}{z}$ and so $z = \frac{1}{w}$.

Therefore $a\left(\frac{1}{w}\right)\left(\frac{\overline{1}}{w}\right)+\overline{b}\left(\frac{1}{w}\right)+b\left(\frac{\overline{1}}{w}\right)+d=0$, which implies $\frac{a}{w\overline{w}}+\frac{\overline{b}}{w}+\frac{b}{\overline{w}}+d=0$ and thus $a+\overline{b}\overline{w}+bw+dw\overline{w}=0$. Let $\alpha,\delta\in\mathbb{R}$ with $\alpha=d$, $\delta=a,\beta=\overline{b}\in\mathbb{C}$.

Therefore $\alpha w\overline{w}+\overline{\beta}w+\beta\overline{w}+\delta=0$ where $|\beta|^2=|\overline{b}|^2=|b^2|>4ad=4\alpha\delta$. So a line through 0 maps to a line through 0, a line not through 0 maps to a circle through 0, a circle not through 0 maps to a circle not through 0, and a circle through 0 maps to a line not through 0.

In conclusion translations, dilations, and reciprocations map circles or lines to circles or lines.

Theorem 7.6: *Let $z_1,z_2,z_3\in\mathbb{C}$ be distinct elements, then there exists a unique $T\in\mathrm{M}$ such that $Tz_1=1$, $Tz_2=0$, and $Tz_3=\infty$.*

Proof:

Note that

$$Tz=\left(\frac{z_1-z_3}{z_1-z_2}\right)\left(\frac{z-z_2}{z-z_3}\right)=\frac{(z_1-z_3)z-z_2(z_1-z_3)}{(z_1-z_2)z-z_3(z_1-z_2)}.$$

Then

$$(z_1-z_3)(-z_3)(z_1-z_2)-(z_1-z_2)(-z_2)(z_1-z_3)=(z_1-z_2)(z_1-z_3)(-z_3+z_2)\neq 0,$$

because z_1,z_2,z_3 are all distinct. Thus it follows that $Tz_1=\left(\frac{z_1-z_3}{z_1-z_2}\right)\left(\frac{z_1-z_2}{z_1-z_3}\right)=1$, $Tz_2=\left(\frac{z_1-z_3}{z_1-z_2}\right)\left(\frac{z_2-z_2}{z_2-z_3}\right)=0$, and $Tz_3=\left(\frac{z_1-z_3}{z_1-z_2}\right)\left(\frac{z_3-z_2}{z_3-z_3}\right)=\infty$. ∎

Theorem 7.7: *Let $z_1,z_2,z_3\in\mathbb{C}$ be distinct elements and let $w_1,w_2,w_3\in\mathbb{C}$ be distinct elements. Then there exists a unique $T\in\mathrm{M}$ such that $Tz_1=w_1$, $Tz_2=w_2$, and $Tz_3=w_3$.*

Proof: By theorem 7.6 there exists a unique $S_1 \in \mathrm{M}$ such that $S_1(z_1)=1$, $S_1(z_2)=0$, and $S_1(z_3)=\infty$. Similarly, there exists a unique $S_2 \in \mathrm{M}$ such that $S_2(w_1)=1$, $S_2(w_2)=0$, and $S_2(w_3)=\infty$. Then let $T=S_2^{-1}\circ S_1 \in \mathrm{M}$. Hence the following hold:

$$T(z_1)=S_2^{-1}(S_1(z_1))=S_2^{-1}(1)=w_1$$

$$T(z_2)=S_2^{-1}(S_1(z_2))=S_2^{-1}(0)=w_2$$

$$T(z_3)=S_2^{-1}(S_1(z_3))=S_2^{-1}(\infty)=w_3. \blacksquare$$

Note that if $T \in \mathrm{M}$. Then $Tz=\dfrac{\alpha z+\beta}{\gamma z+\delta}, \alpha\delta-\beta\gamma \neq 0$, and

$$T'z=\frac{(\gamma z+\delta)(\alpha)-(\alpha z+\beta)(\gamma)}{(\gamma z+\delta)^2}=\frac{\alpha\gamma z+\alpha\delta-\alpha\gamma z-\beta\gamma}{(\gamma z+\delta)^2}=\frac{\alpha\delta-\beta\gamma}{(\gamma z+\delta)^2}\neq 0.$$

Therefore Tz is an analytic mapping where $T'z \neq 0$ and such mappings are conformal. This is desirable because it implies that Möbius transformations mappings that preserve angles.

CHAPTER 8: HYPERBOLIC ISOMETRIES

We now turn our attention back to the hyperbolic plane and will describe hyperbolic isometries with complex numbers. We know that reflections about a vertical geodesic are isometries in the hyperbolic plane, and hence horizontal translations are also isometries of the hyperbolic plane. Clearly either $f(z) = z + r$ or $f(z) = \overline{z} + r$ where $r \in \mathbb{R}$, are complex descriptions of these isometries.

We also know that Euclidean inversions centered on the x-axis are hyperbolic isometries, in particular they are reflections. If $z' = I_{0,K}(z)$, then z' lies on the line through the origin and z, thus it follows directly that $\arg(z') = \arg(z)$ and $|z||z'| = k^2$. Since $\arg\left(\frac{k^2}{\overline{z}}\right) = \arg(z)$ and $|z|\left|\frac{k^2}{\overline{z}}\right| = k^2$, then $z' = I_{0,K}(z) = \frac{k^2}{\overline{z}}$. So we can describe any inversion as $I_{A,K}(z) = \tau_{0A} \circ I_{0,k} \circ \tau_{A0} = \frac{k^2}{\overline{z} - a} + a$. Since hyperbolic isometries are compositions of Euclidean inversions then we can characterize hyperbolic isometries as described above.

Since we know that every hyperbolic isometry is the composition of several hyperbolic reflections, then we can describe all hyperbolic isometry as explicit expressions. From this, the following theorem is developed.

Theorem 8.1: *The orientation preserving hyperbolic isometries in the upper half-plane model are of the form* $f(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$ *, where* $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ *and* $\alpha\delta - \beta\gamma > 0$ *and* $\overline{f}(z) = \frac{-\alpha\overline{z} + \beta}{-\gamma\overline{z} + \delta}$ *, where* $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ *and* $\alpha\delta - \beta\gamma > 0$ *are the orientating reversing isometries .*

Proof: Clearly the horizontal translations have the form $\frac{1z+r}{0z+1}$, the reflections in the straight geodesics have the form $\frac{1(-\bar{z})+r}{0(-\bar{z})+1}$, and reflections over the bowed geodesics have the form $\frac{k^2}{\bar{z}-a}+a=\frac{-a(-\bar{z})+(k^2-a^2)}{-(-\bar{z})-a}$. In each case $\alpha\delta-\beta\gamma>0$. We know that the composition of Möbius transformations, is again a Möbius transformation. The relation $\alpha\delta-\beta\gamma>0$ still holds. So let $f(z)=\frac{\alpha z+\beta}{\gamma z+\delta}$ where $\alpha\delta-\beta\gamma>0$ and let $g(z)=\frac{\alpha'(-\bar{z})+\beta'}{\gamma'(-\bar{z})+\delta'}$ where $\alpha'\delta'-\beta'\gamma'>0$.

Then

$$(f\circ g)(z)=\frac{\alpha\frac{\alpha'(-\bar{z})+\beta'}{\gamma'(-\bar{z})+\delta'}+\beta}{\gamma\frac{\alpha'(-\bar{z})+\beta'}{\gamma'(-\bar{z})+\delta'}+\delta}=\frac{(\alpha\alpha'+\beta\gamma')(-\bar{z})+(\alpha\beta'+\beta\delta')}{(\gamma\alpha'+\delta\gamma')(-\bar{z})+(\gamma\beta'+\delta\delta')},$$

and

$$(\alpha\alpha'+\beta\gamma')(\gamma\beta'+\delta\delta')-(\alpha\beta'+\beta\delta')(\gamma\alpha'+\delta\gamma')$$

$$=\alpha\alpha'\gamma\beta'+\beta\gamma'\gamma\beta'-\alpha\beta'\delta\gamma'-\beta\delta'\gamma\alpha'$$

$$=(\alpha\delta-\beta\gamma)(\alpha'\delta'-\beta'\gamma')>0.$$

Also

$$(f\circ f)(z)=\frac{\alpha\frac{\alpha z+\beta}{\gamma z+\delta}+\beta}{\gamma\frac{\alpha z+\beta}{\gamma z+\delta}+\delta}=\frac{(\alpha\alpha+\beta\gamma)z+(\beta\alpha+\beta\delta)}{(\gamma\alpha+\gamma\delta)z+(\gamma\beta+\delta\delta)},$$

and

$$(\alpha\alpha+\beta\gamma)(\gamma\beta+\delta\delta)-(\alpha\beta+\beta\delta)(\gamma\alpha+\delta\gamma)$$

$$=\alpha\alpha\delta\delta-\alpha\delta\beta\gamma-\alpha\delta\beta\gamma\alpha+\beta\beta\gamma\gamma$$

$$=(\alpha\delta-\beta\gamma)^2>0.$$

Finally,

$$(g\circ g)(z)=\frac{\alpha'(-1)\left(\dfrac{\alpha'(-\overline{z})+\beta'}{\gamma'(-\overline{z})+\delta'}\right)+\beta'}{\gamma'(-1)\left(\dfrac{\alpha'(-\overline{z})+\beta'}{\gamma'(-\overline{z})+\delta'}\right)+\delta'}=\frac{(\alpha'\alpha'-\beta'\gamma')z+(\beta'\delta'-\alpha'\beta')}{(\gamma'\alpha'-\delta'\gamma')z+(\delta'\delta'-\gamma'\beta')},$$

and

$$\text{s}(\alpha'\alpha'-\beta'\gamma')(\delta'\delta'-\gamma'\beta')-(\gamma'\alpha'-\delta'\gamma')(\beta'\delta'-\alpha'\beta')$$

$$=\alpha'\alpha'\delta'\delta'-\alpha'\delta'\gamma'\beta'-\alpha'\delta'\gamma'\beta'+\beta'\beta'\gamma'\gamma'$$

$$=(\alpha'\delta'-\beta'\gamma')^2>0.$$

Since every hyperbolic isometry is the composition of some hyperbolic reflection, then it follows that all hyperbolic isometries have either the form of $f(z)$ or $g(z)$.

Conversely, consider $f(z)=\dfrac{\alpha z+\beta}{\gamma z+\delta}$ where $\alpha\delta-\beta\gamma>0$ and $\gamma\neq 0$. Then through some algebraic manipulations it can easily be shown that

$$f(z)=\frac{\alpha z+\beta}{\gamma z+\delta}=-\left[\frac{\dfrac{(\alpha\delta-\beta\gamma)}{\gamma^2}}{\overline{z}-\left(-\dfrac{\delta}{\gamma}\right)}+\left(-\frac{\delta}{\gamma}\right)\right]+\left(\frac{\alpha-\delta}{\gamma}\right).$$

Therefore $f(z)$ is the composition of the inversion $I_{\left(\frac{-\delta}{\gamma},0\right),\frac{\sqrt{\alpha\delta-\beta\gamma}}{|\gamma|}}$ with the reflection in the straight geodesic above the point $\left(\frac{\alpha-\delta}{2\gamma},0\right)$, which are both hyperbolic isometries. Similarly if $\gamma=0$ and $\delta\neq 0$ then $f(z)=\frac{\alpha}{\delta}z+\frac{\beta}{\delta}$. Thus $f(z)$ is the composition of a dilation with a horizontal translation, which are both hyperbolic isometries.

Now suppose that $g(z)=\frac{\alpha(-\bar{z})+\beta}{\gamma(-\bar{z})+\delta}$ where $\alpha\delta-\beta\gamma>0$ and $\gamma\neq 0$. Then with some algebraic manipulations it can easily be shown that

$$g(z)=\frac{\alpha(-\bar{z})+\beta}{\gamma(-\bar{z})+\delta}=\frac{\frac{\alpha\delta-\beta\gamma}{\gamma^2}}{\bar{z}-\frac{\delta}{\gamma}}+\frac{\delta}{\gamma}+\frac{\alpha-\delta}{\gamma}.$$

So $g(z)$ is the composition of an inversion in a bowed geodesic with a horizontal translation, which are both hyperbolic isometries. Similarly, if $\gamma=0$ and $\delta\neq 0$, then it can be shown that $g(z)=-\frac{\alpha}{\delta}\bar{z}+\frac{\beta}{\delta}$. Thus $g(z)$ is the composition of a reflection over the y-axis, a dilation, and a horizontal translation. ■

CHAPTER 9: POINCARE UNIT DISK

There is an isometry from the upper half plane to the unit disk and it happens to be a Möbius transformation. Consider the mapping that fixes both 1 and -1 but sends 0 to $-i$. Note that Möbius transformations preserve cross ratios, which allows us to easily determine the Möbius transformations given the mapping of three points. Therefore

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}.$$

Let $z_1=1,\ z_2=0,\ z_3=-1,\ w_1=1,\ w_2=-i$, and $w_3=-1$.

By the cross ratio,

$$\frac{(w-1)(-i+1)}{(w+1)(-i-1)}=\frac{(z-1)}{(-z-1)}.$$

Therefore

$$(w-1)(-i+1)(-z-1)=(w+1)(-i-1)(z-1).$$

So $izw+z-w-i=-izw-z+w+i$ and thus $-2izw+2w=2z-2i$. Finally, $w=U(z)=\dfrac{z-i}{-iz+1}=\dfrac{iz+1}{z+i}$ is the isometry from the upper half plane model into the disk model of the hyperbolic plane.

Now consider $V(z)=\dfrac{iz-1}{-z+i}$.

Note that

$$U(V(z))=\frac{i\left(\dfrac{iz-1}{-z+i}\right)+1}{\left(\dfrac{iz-1}{-z+i}\right)+i}=\frac{i(iz-1)+(-z+i)}{(iz-1)+i(-z+i)}=\frac{-2z}{-2}=z$$

and

$$V(U(z)) = \frac{i\left(\frac{iz+1}{z+i}\right)-1}{-\left(\frac{iz+1}{z+i}\right)+i} = \frac{i(iz+1)-(z+i)}{-(iz+1)+i(z+i)} = \frac{-2z}{-2} = z.$$

Therefore $V(z) = \frac{iz-1}{-z+i}$ is the inverse of $U(z)$, and hence there is a one to one correspondence between the upper half plane and the unit disk. Later we will show that $U(z) = \frac{iz+1}{z+i}$ is an isometry. Since isometries map geodesics to geodesics, then we know that the mapping $U(z) = \frac{iz+1}{z+i}$ maps the geodesics of the upper half plane to geodesics of the unit disk. Since Möbius transformations are conformal and map circles or lines to circle or lines, then we can explicitly describe the geodesics in the unit disk. One type of geodesic in the upper half plane are vertical lines that are perpendicular to the x-axis. It can be shown that these geodesics are mapped to the diameters of the unit disk. Also circles perpendicular to the x-axis are geodesics in the upper half plane, and so these will map to circular arcs that are orthogonal to the unit circle. Thus it follows that the geodesics of the unit disk are the diameters and circular arcs orthogonal to the unit circle.

Now let $w = x + iy$ be a point in the interior of the unit disk and let $z = u + iv$ be the corresponding point in the upper half plane. Clearly $z = V(w) = \frac{iw-1}{-w+i}$ and therefore

$$u+iv = \frac{i(x+iy)-1}{-(x+iy)+i} = \frac{(y+1)-ix}{x+i(y-1)} = \frac{(y+1)-ix}{x+i(y-1)} \cdot \frac{x-i(y-1)}{x-i(y-1)} = \frac{2x+i\left(1-x^2-y^2\right)}{x^2+(y-1)^2}.$$

Thus, $u = \frac{2x}{x^2+(y-1)^2}$ and $v = \frac{1-x^2-y^2}{x^2+(y-1)^2}$.

So

$$\frac{\partial u}{\partial x}=\frac{\left(x^2+(y-1)^2\right)(2)-2x(2x)}{\left(x^2+(y-1)^2\right)^2}=\frac{-2x^2+2(y-1)^2}{\left(x^2+(y-1)^2\right)^2}$$

$$\frac{\partial u}{\partial y}=\frac{\left(x^2+(y-1)^2\right)(0)-2x(2(y-1))}{\left(x^2+(y-1)^2\right)^2}=\frac{-4x(y-1)}{\left(x^2+(y-1)^2\right)^2}$$

$$\frac{\partial v}{\partial x}=\frac{\left(x^2+(y-1)^2\right)(-2x)-\left(1-x^2-y^2\right)(2x)}{\left(x^2+(y-1)^2\right)^2}=\frac{-4x(y-1)}{\left(x^2+(y-1)^2\right)^2}$$

$$\frac{\partial v}{\partial y}=\frac{\left(x^2+(y-1)^2\right)(-2y)-\left(1-x^2-y^2\right)(2(y-1))}{\left(x^2+(y-1)^2\right)^2}=\frac{-2x^2+2(y-1)^2}{\left(x^2+(y-1)^2\right)^2}.$$

Recall $du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$ and $dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy$ and so

$$du^2+dv^2=\left(\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy\right)^2+\left(\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy\right)^2$$

$$=\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right]dx^2+2\left(\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y}\right)dxdy+\left[\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]dy^2.$$

Note that since Möbius transformations are analytic, then the Cauchy-Riemann Equations hold; thus

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\text{ and }\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.$$

Therefore

$$\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y}=0 \text{ and } \left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right]=\left[\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right].$$

Therefore

$$du^2+dv^2=\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right]\left(dx^2+dy^2\right)=\frac{4\left[x^2-(y-1)^2\right]^2+16x^2\left(y-1^2\right)}{\left[x^2+(y-1)^2\right]^4}\left(dx^2+dy^2\right)$$

$$=\frac{4x^4-8x^2(y-1)^2+4(y-1)^4+16x^2(y-1)^2}{\left[x^2+(y-1)^2\right]^4}\left(dx^2+dy^2\right)=\frac{4\left[x^2+(y-1)^2\right]^2}{\left[x^2+(y-1)^2\right]^4}\left(dx^2+dy^2\right)$$

$$=\frac{4\left(dx^2+dy^2\right)}{\left[x^2+(y-1)^2\right]^2}.$$

Hence $\dfrac{du^2+dv^2}{v^2}=\dfrac{\dfrac{4\left(dx^2+dy^2\right)}{\left[x^2+(y-1)^2\right]^2}}{\left(\dfrac{1-x^2-y^2}{x^2+(y-1)^2}\right)^2}=\dfrac{4\left(dx^2+dy^2\right)}{\left(1-x^2-y^2\right)^2}$.

The metric in the unit disk model of the hyperbolic plane is $\dfrac{4\left(dx^2+dy^2\right)}{\left(1-x^2-y^2\right)^2}$.

Let $\gamma(t)=\left(x(t),y(t)\right)$, $a\le t\le b$ be a parameterized curve in the unit disk. Since we showed there was an isometry from the upper half plane to the unit disk, then the length of $\gamma(t)$ is equal to some $\gamma^*(t)=h(\gamma(t))$ in the upper half plane. The curve γ^* can be parameterized as $\left(u(t),v(t)\right)$, $a\le t\le b$ where u and v are as given above. Then the hyperbolic length is $\int_{\gamma^*} \frac{\sqrt{du^2+dv^2}}{v} = \int_{\gamma} \frac{2\sqrt{dx^2+dy^2}}{1-x^2-y^2}$. So the length of a curve in the unit disk model is $\int_{\gamma} \frac{2\sqrt{dx^2+dy^2}}{1-x^2-y^2}$.

In the unit disk model, $E=\frac{4}{\left(1-u^2-v^2\right)^2}$, $F=0$, and $G=\frac{4}{\left(1-u^2-v^2\right)^2}$, and hence we will use the equation $K=\frac{-1}{2\sqrt{EG}}\left\{\left(\frac{E_v}{\sqrt{EG}}\right)_v+\left(\frac{G_u}{\sqrt{EG}}\right)_u\right\}$ to find the curvature of the unit disk model. Note that $E_v=\frac{16v}{\left(1-u^2-v^2\right)^3}$ and $G_u=\frac{16u}{\left(1-u^2-v^2\right)^3}$.

Therefore

$$K=\frac{-1}{2\sqrt{\frac{16}{\left(1-u^2-v^2\right)^4}}}\left\{\left(\frac{\frac{16v}{\left(1-u^2-v^2\right)^3}}{\sqrt{\frac{16}{\left(1-u^2-v^2\right)^4}}}\right)_v+\left(\frac{\frac{16u}{\left(1-u^2-v^2\right)^3}}{\sqrt{\frac{16}{\left(1-u^2-v^2\right)^4}}}\right)_u\right\}$$

$$=\frac{-1}{2\frac{4}{\left(1-u^2-v^2\right)^2}}\left\{\left(\frac{\frac{16v}{\left(1-u^2-v^2\right)^3}}{\frac{4}{\left(1-u^2-v^2\right)^2}}\right)_v+\left(\frac{\frac{16u}{\left(1-u^2-v^2\right)^3}}{\frac{4}{\left(1-u^2-v^2\right)^2}}\right)_u\right\}$$

$$=\frac{-1\left(1-u^2-v^2\right)^2}{8}\left\{\left(\frac{4v}{\left(1-u^2-v^2\right)}\right)_v+\left(\frac{4u}{\left(1-u^2-v^2\right)}\right)_u\right\}$$

$$=\frac{-1\left(1-u^2-v^2\right)^2}{8}\left\{\left(\frac{4\left(1-u^2-v^2\right)+8v^2}{\left(1-u^2-v^2\right)^2}\right)+\left(\frac{4\left(1-u^2-v^2\right)+8u^2}{\left(1-u^2-v^2\right)^2}\right)\right\}$$

$$= \frac{-1\left(1-u^2-v^2\right)^2}{8}\left(\frac{8}{\left(1-u^2-v^2\right)^2}\right) = -1.$$

Thus the unit disk has constant curvature -1. This is what we hoped to get, since the curvature of the upper half plane and the curvature of the pseudosphere is -1.

We can describe isometries of the unit disk as linear fractional transformations just as we did in the upper half plane. The isometries of the unit disk are obtained from isometries of the upper half plane. Note that for two points P and Q in the unit disk, we can denote $j(P,Q)$ as the distance between the two points P and Q in the unit disk and denote $h(V(P),V(Q))$ as the distance between two points $V(P)$ and $V(Q)$ in the upper half plane. Since $V(z)$ is the inverse of the isometric mapping from the upper half plane into the disk, then clearly $j(P,Q) = h(V(P),V(Q))$. If $f(z)$ is an isometry of the upper half plane, then $w(z) = U(z) \circ f(z) \circ V(z)$ is a hyperbolic isometry of the unit disk where $U(z) = \frac{iz+1}{z+i}$ and $V(z) = \frac{iz-1}{-z+i}$.

Thus

$$j(w(P),w(Q)) = j\left(U\left(f\left(V(P)\right)\right),\mathrm{U}\left(f\left(V(Q)\right)\right)\right)$$

$$= h\left(V\left(U\left(f\left(V(P)\right)\right)\right),\mathrm{V}\left(\mathrm{U}\left(f\left(V(Q)\right)\right)\right)\right) = h\left(f\left(V(P)\right),f\left(V(Q)\right)\right)$$

$$= h\left(V(P),V(Q)\right) = j(P,Q).$$

Theorem 9.1: *The isometries of the unit disk are the transformations of the forms* $g(z) = \frac{\alpha z+\overline{\beta}}{\beta z+\overline{\alpha}}$ *and* $k(z) = \frac{-\alpha\overline{z}+\overline{\beta}}{-\beta\overline{z}+\overline{\alpha}}$, *where* α *and* β *are complex numbers such that* $|\alpha| > |\beta|$.

Proof: Let $f(z)=\dfrac{az+b}{cz+d}$ be any Möbius isometry of the upper half plane and so $a,b,c,d\in\mathbb{R}$ and $ad-bc>0$.

So

$$w(z)=U(z)\circ f(z)\circ V(z)=\frac{iz+1}{z+i}\circ\frac{az+b}{cz+d}\circ\frac{iz-1}{-z+i}$$

$$=\frac{iz+1}{z+i}\circ\frac{(ai-b)z+(-a+ib)}{(ci-d)z+(-c+id)}=\frac{\left[(-a-d)+i(c-b)\right]z+\left[(-b-c)+i(-a+d)\right]}{\left[(-b-c)+i(a-d)\right]z+\left[(-a-d)+i(-c+b)\right]}.$$

Let $\alpha=\left[(-a-d)+i(c-b)\right]$ and $\beta=\left[(-b-c)+i(a-d)\right]$.

Then

$$w(z)=U(z)\circ f(z)\circ V(z)=\frac{\alpha z+\bar{\beta}}{\beta z+\bar{\alpha}}.$$

Also

$$|\alpha|^2-|\beta|^2=\left|\left[(-a-d)+i(c-b)\right]\right|^2-\left|\left[(-b-c)+i(a-d)\right]\right|^2=4(ad-bc)>0.$$

Therefore $|\alpha|^2>|\beta|^2$, and hence $|\alpha|>|\beta|$.

Conversely, suppose $w(z)=\dfrac{\alpha z+\bar{\beta}}{\beta z+\bar{\alpha}}$ such that $|\alpha|>|\beta|$.

Then we can set

$$f(z)=V(z)\circ w(z)\circ U(z)=\frac{iz-1}{-z+i}\circ\frac{\alpha z+\bar{\beta}}{\beta z+\bar{\alpha}}\circ\frac{iz+1}{z+i}=\frac{az+b}{cz+d},$$

where

$$a = -(\alpha + \overline{\alpha}) - i(\beta - \overline{\beta})$$

$$b = -(\beta + \overline{\beta}) + i(\alpha - \overline{\alpha})$$

$$c = -(\beta + \overline{\beta}) + i(\alpha - \overline{\alpha})$$

$$d = -(\alpha + \overline{\alpha}) + i(\beta - \overline{\beta}).$$

Note that $\alpha + \overline{\alpha} = 2\operatorname{Re}(\alpha)$, $\beta + \overline{\beta} = 2\operatorname{Re}(\beta)$, $\alpha - \overline{\alpha} = 2i\operatorname{Im}(\alpha)$, and $\beta + \overline{\beta} = 2i\operatorname{Im}(\beta)$, and so $a, b, c, d \in \mathbb{R}$.

Also

$$ad - bc = \left[(\alpha + \overline{\alpha})^2 + (\beta - \overline{\beta})^2\right] - \left[(\beta + \overline{\beta})^2 + (\alpha - \overline{\alpha})^2\right]$$

$$= 4(\alpha\overline{\alpha} - \beta\overline{\beta}) = 4\left(|\alpha|^2 - |\beta|^2\right) > 0.$$

Therefore $f(z)$ is an isometry of the upper half plane.

Since $w(z) = U \circ f \circ V = U \circ (V \circ w \circ U) \circ V = (U \circ V) \circ w \circ (U \circ V) = w$, then if follows that $w(z) = \dfrac{\alpha z + \overline{\beta}}{\beta z + \overline{\alpha}}$ is a Mobius isometry of the upper half plane.

In a similar way it is shown that $k(z) = \dfrac{-\alpha\overline{z} + \overline{\beta}}{-\beta\overline{z} + \overline{\alpha}}$ is also an isometry of the unit disk. ■

CHAPTER 10: HYPERBOLIC XY-PLANE

There is another representation of the hyperbolic plane that resembles the entire xy-plane. We will denote this model by $\mathbb{H}$. Let $S = \mathbb{R}^2$ be a plane with coordinates (u,v) and define $E = \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle = 1$, $F = \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\rangle = 0$, and $G = \left\langle \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \right\rangle = e^{2u}$. Using the fact that $K = -\frac{1}{2\sqrt{EG}}\left\{ \left(\frac{E_v}{\sqrt{EG}} \right)_v + \left(\frac{G_u}{\sqrt{EG}} \right)_u \right\}$ and after some algebraic calculations we will see that this model also has constant curvature of -1. Note that $E_v = 0$ and $G_u = 2e^{2u}$ and so $K = -\frac{1}{2e^u}\left(\frac{2e^{2u}}{e^u} \right)_u = -\frac{1}{2e^u}\left(2e^u\right) = -1$.

Now consider the geodesics in this model. To find the geodesics define $\phi : \mathrm{H} \to \mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$ by $\phi(u,v) = (v, e^{-u})$. Note that ϕ is differentiable and since $y > 0$ then it has a differentiable inverse. Therefore ϕ is a diffeomorphism and we can induce an inner product $\mathbb{R}^2_+$ by $\langle d\phi(w_1), d\phi(w_2) \rangle_{\phi(q)} = \langle w_1, w_2 \rangle_q$. Note that $\frac{\partial}{\partial x} = \frac{\partial}{\partial v}$ and $\frac{\partial}{\partial y} = -e^u \frac{\partial}{\partial u}$. Thus if follows from the inner product that $E = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \right\rangle = \left\langle \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \right\rangle = e^{2u} = \frac{1}{y^2}$, $F = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle = -e^u \left\langle \frac{\partial}{\partial v}, \frac{\partial}{\partial u} \right\rangle = 0$, and $G = \left\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \right\rangle = e^{2u} \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle = \frac{1}{y^2}$. Therefore $\mathbb{R}^2_+$ is isometric to $\mathbb{H}$ with the given inner product but $\mathbb{R}^2_+$ is the upper half plane. So $\mathbb{H}$ is also isometric to $\mathbb{H}^2$.

CHAPTER 11: CROCHETING THE HYPERBOLIC PLANE

The hyperbolic models are illuminating but provide a distorted look through a rather strange lens and do not reveal what a hyperbolic surface looks like in our world. The search for a realistic model was in full force until about the nineteenth century. In 1901, David Hilbert proved that it is not possible to embed the entire hyperbolic plane in $\mathbb{R}^3$. In fact, he proved that it cannot even be immersed in $\mathbb{R}^3$ with self-intersections. But it left some mathematicians with an interest in coming up with models for hyperbolic surfaces. In $\mathbb{R}^2$, the plane can be tessellated with six triangles meeting at each vertex. William Thurston found a low-tech approach to make a hyperbolic model, in the 1970's. Thurston used paper and scissors and came up with a model made by gluing triangles together. His idea was to approximate the hyperbolic plane by having seven triangles meet at each vertex. A colleague of his at Cornell, Daina Taimina, used Thurston's model with her students. She had a hard time with the model because it was very fragile; it would fall apart easily. She hated gluing the pieces together and came up with the idea to crochet a model of the hyperbolic plane instead. Her idea was simple: she started with a row of stitches and then would add a fixed amount of stitches each row after that. She hoped that this would create a piece of fabric that became wider and wider. After some trial and error she was able to create a model that has given important insight into an abstract area of math. She was able to trace straight lines in and out of the expanding flaps and realized she could trace parallel lines that diverged. The models are not perfect and are only a rough approximation of what in theory should be a smooth surface. Overall, the crochet model gives students a surface to hold and visually experience the hyperbolic plane. In theory, if a piece of hyperbolic crochet had an infinite number of stitches, it would be possible to live on that surface and walk in some direction without coming to an edge.

Hyperbolic surfaces that surround a three-dimensional region increase the surface area to volume ratio. Thus hyperbolic surfaces are favored by some plants and marine organisms. The most commonly known organism that does this is coral. Coral needs a large surface to absorb nutrition and thus grow in a hyperbolic way.

Hyperbolic geometry was not always accepted as it is today. Hyperbolic geometry at one point was considered to go against a sense of reality and many mathematicians did not accept it as a geometry. Henri Poincare said it best, "one geometry cannot be more true than another; it can only be more convenient." (Bellos) Therefore every surface has its own geometry and, for any practical purpose, we should choose the one that applies best.

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