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Mathematical Explorations of Card Tricks

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Introduction

In this project, we explore various mathematical topics as they apply to an assortment of card tricks. We will focus on an examination of theorems applied to the manipulation of cards in an attempt to prove why certain card tricks work. These theorems utilize abstract algebra, probability, number theory, and combinatorics. Many tricks can be explained this way, instead of singularly by sleight of hand or other “magical” methods. We will rigorously prove the theorems and principles that explain these concepts, focusing primarily on the card tricks and examples presented in *Mathematical Card Magic* by Colm Mulcahy (2013).

While the study of pure, theoretical mathematics is very interesting, it is also helpful to see how the subject can be applied. Applications of math are prevalent and can be very practical, such as in engineering and economics. The field of mathematical magic combines the academic aspect of mathematics with the entertainment of card tricks and magic. Thus, this topic shows that understanding mathematics can result in something that even non-mathematicians can enjoy. For a future teacher, this project could fit in well with a classroom talk in order to help demonstrate how math can be fun for kids. The required sleight of hand and quick thinking required by the magic aspect can also prepare the speaker for future work as a teacher.

Luckily, the topic of math as it relates to card tricks has been well researched. The substance of this project involves examining a book by Colm Mulcahy, a well-known expert in the area of mathematical card tricks. According to Mulcahy, work in mathematical magic started in the early twentieth century (Mulcahy xiii). However, math was a recreational activity possibly as far back as ancient Mesopotamian civilizations and the early Egyptian empires (Merzback 7). In the middle of the twentieth century, Martin Gardner released large amounts of data on mathematical magic, and this is when the field truly began to flourish (Mulcahy xiii). Mulcahy’s book goes into depth both on the magical and mathematical aspects of the tricks. While he presents theorems and principles, he does not prove the results, or at least not to the extent that would be expected in an upper level math class. Those rigorous proofs are the primary objective of this project. *Mathematical Card Magic* has such a wide range of examples that it has not been necessary to pull tricks from any other sources in this one semester project.

We start with some mathematical and card related principles which will eliminate redundancy and possible confusion about vocabulary when we move into the explanation of specific tricks. In each later section, we will describe a trick and explain it mathematically. When selecting which tricks to include in this paper, we chose those that have interesting mathematical foundations. As a secondary consideration we thought of practicality of performance, based on the mental mathematics and advanced sleight of hand necessary for performance. We conclude the paper with possible further investigations related to this project.

0. General Principles

Before examining card tricks in the context of their mathematical basis, it is important to have a basic understanding of some fundamental principles. These principles are a mixture of facts related to math and/or cards. For example, we use the convention that a full deck has 52 cards with four suits (hearts, diamonds, clubs, and spades).

First, the shuffling of cards is clearly of high importance in card tricks. Unless otherwise stated, we assume that any manner of shuffling is sufficient. Some tricks, such as that in V.b, explicitly call for riffle shuffling. Riffle shuffling (or "riffling") a deck of cards means dividing it into two packets, bending the cards with each thumb, and releasing the cards so that the cards intermix in a single pile. This type of shuffling is often done in tandem with "bridging," which simply re-bends the cards to maintain their shape. An explanation of riffle shuffling is found on page 3 of Mulcahy and many videos online demonstrate this technique. At other times fake shuffling will be required (such as in trick III.a). There are many different ways to pretend to shuffle a deck, with varying degrees of difficulty. Mulcahy explains different ways to fake shuffle in his section on shuffling (1-13).

Sometimes, either instead of or in tandem with fake shuffling, the magician must know the order of the cards. We use the mnemonic word "CHaSeD" to describe a deck ordering which seems random, but is easy to remember. The capital letters in the word CHaSeD refer to the four suits and the order of the letters indicates the order of the suits in the deck. Specifically, in a packet of CHaSeD cards, the suits are in this order: Clubs, Hearts, Spades, Diamonds. Sometimes these suits allow us to designate magnitude also, such as clubs being less than hearts, which is less than spades, which is less than diamonds. Using this, the magician can remember which cards were present in the packet. For example, if an ace, two, three, and four are used, it is easier to memorize their suits if they are in CHaSeD order than in another order (Mulcahy 13).

The dealing of cards is of special importance. Many tricks (especially the tricks in Section I) are completely based on the way the cards are dealt. Hence, it is useful to note that when cards are dealt from the top of a deck into a new stack of cards, the order of the cards is reversed. Conversely, when cards are dealt from the bottom of the deck, the cards remain in the same order. The former of these realizations is especially important for the next section of tricks, which is based on COATing. COAT stands for **C**ount **O**ut **A**nd **T**ransfer and refers to counting out *k* cards from the top of an *n* sized packet and transferring the resulting stack of *k* cards to the bottom of the packet (Mulcahy 35). As referenced above, the *k* cards dealt from the top will be in reverse order at the bottom of the deck. Mulcahy uses the term *overCOAT* to refer to this process when $k \geq \frac{n}{2}$. We will simply use the term COAT, and indicate the instances in which $k \geq \frac{n}{2}$ is required.

It is important to know how to count the number of cards in an ordered sequence. When we subtract two whole numbers, we are really counting the number of one-unit gaps between those numbers on a number line, as opposed to counting the numbers themselves. We use this principle when determining the distance between two cards in a deck. For example $8-5=3$, so there are two cards in the packet between the card in position five and the card in position eight, and there are four cards starting with position five through position eight. Hence, in general, there are $n-k+1$ cards in positions *k* through *n*. This will be especially useful for the section on COATs.

Another counting idea that is useful when doing card tricks is modular arithmetic. Modular arithmetic is sometimes referred to as "clock arithmetic," because it functions similarly to the fact that two o'clock is four hours after ten o'clock: on a 12-hour clock, $10+4=2$. In arithmetic modulo *n*, the sum of two numbers is equivalent to the remainder when the sum is divided by *n*. For example $10+4=14$ has a remainder of 2 when divided by 12, thus $14=2(\text{mod}12)$.

Finally, the Pigeonhole Principle surfaces several times in this paper. This principle states that if *n* items occupy *k* spaces and $n>k$, then clearly at least one space must be occupied by at least two items. A discussion of this principle appears in most discrete mathematics books.

I. COATs

We begin our discussion of card tricks by looking at two tricks that rely on properties of the COAT procedure.

I.a. Four Scoop Triple Revelation. This trick is a combination of the tricks “Three Scoop Miracle” and “Triple Revelation” presented by Mulcahy (25, 37).

Description of the Trick

Start by having three volunteers each pick one card at random from a deck that is approximately 13 cards. They should look at and memorize their cards. Have them place their cards on the top of the deck. Ask a volunteer for his favorite ice cream flavor. If necessary, ask the volunteer to adjust the name of the flavor so that it is long enough, i.e. more than half the deck size (such as changing “mint” to “mint chip” or “peppermint”). Tell the audience that you are going to make a sundae and need to scoop the ice cream. As a demonstration, COAT the cards (as described in the General Principles section) while spelling the ice cream flavor – one card per letter. Then instruct each of the volunteers, in turn, to COAT the cards as described above. Then hand the deck to the last volunteer who placed his card on top. Have him reveal the top card and notice it is his card. Then hand the deck to the second volunteer and do the same. Finally, do this with the remaining volunteer.

Mathematical Analysis

This trick is clearly an application of COATing. We are interested in the top three cards and their movements throughout the deck. Let the deck consist of *n* cards and the number of letters in the flavor be *k*, with $\frac{n}{2} \leq k \leq n$.

Suppose the volunteers choose cards *x*, *y*, and *z*, respectively. The cards *x*, *y*, and *z* begin on top of the deck in this order. After the first (demonstration) COAT, the three bottom cards are *z*, *y*, and *x*, in this order.

The "Save at Least 50% Principle" below demonstrates that after 3 COATs, the cards that are originally on the bottom of the deck move to the top of the deck, but in reverse order. Therefore, after the three COATs performed by the volunteers, cards *x*, *y*, and *z* are again on the top of the deck, in their original order.

Save at Least 50% Principle: If *k* cards from *n* are COATed three times, then provided that $k \geq \frac{n}{2}$, the original bottom *k* cards become the top *k* cards, in reverse order. That is to say, three COATs preserve at least half the packet – the bottom half – only in reversed order, at the top.

Proof: Let the deck have initial order $a_1, a_2, \ldots, a_n$, where a_1 is the top card (dealt first). Let $\frac{n}{2} \leq k \leq n$. Then, after one COAT of *k* cards, the ordering of the deck is

$$a_{k+1}, a_{k+2}, \ldots, a_n, a_k, a_{k-1}, \ldots, a_1 .$$

Note that $\{a_{k+1}, a_{k+2}, \ldots, a_n\}$ contains $n-k$ cards and $n-k \leq k$. So these cards, possibly with some additional cards, will be COATed in the next iteration. Specifically, this next COAT moves $k-(n-k) = 2k-n$ cards in addition to the cards $a_{k+1}, \ldots, a_n$. Thus, the last COATed card will be a_i, where

$$i = \begin{cases} k-(2k-n)+1 = (n+1)-k & \text{if} \quad n-k < k \\ n & \text{if} \quad n-k = k \end{cases} .$$

Therefore, after the second COAT the ordering of the deck is

$$a_{n-k}, a_{n-k-1}, \ldots, a_1, a_{(n+1)-k}, a_{(n+2)-k}, \ldots, a_k, a_n, a_{n-1}, \ldots, a_{k+1} .$$

Notice that, as unordered sets,

$$\left\{a_{n-k}, a_{n-k-1} \ldots, a_1, a_{(n+1)-k}, a_{(n+2)-k} \ldots, a_k\right\} = \left\{a_1, a_2, \ldots, a_{n-k}, a_{n-k+1}, \ldots, a_k\right\}$$

and thus the sequence $a_{n-k}, \ldots, a_k$ in the twice-COATed deck contains *k* cards. Therefore, the third COAT of *k* cards results in this ordering of the deck:

$$a_n, a_{n-1}, \ldots, a_{k+1}, a_k, \ldots, a_{(n+1)-k}, a_1, a_2, \ldots, a_{n-k}.$$

Finally, since the sequence $\left\{a_1, a_2, \ldots, a_{n-k}\right\}$ contains $n-k$ cards, it follows that the sequence $\left\{a_n, a_{n-1}, \ldots, a_{(n+1)-k}\right\}$ contains *k* cards. This second sequence is clearly at the top of the deck and contains the cards that were originally on the bottom of the deck, but in reversed order.■

I.b. Ace Combination. This trick is the trick "Ace Combination" presented by Mulcahy but with a slight variation (41-42).

Description of the Trick

Have a volunteer choose a three digit number, *abc*, that will be the combination of a safe. Indicate that the keypad only contains prime and composite numbers, so 0 and 1 are not available. Use the first two digits, *a* and *b,* to count out a packet of $2a+b$ cards. Have the volunteer COAT $a+b$ cards *c* times, where *c* is the third digit, before handing the combined packet back to the magician. The magician puts this packet out of his and the audience's sight (behind his back or under a table), manipulates the cards, and then shuffles the packet. The magician now produces the shuffled packet to reveal all four aces overturned while the other cards are still face down.[1]

[1] The magician also has the option of simply revealing the aces, in the case of a less proficient magician.

Mathematical Analysis and Trick Explanation

Before beginning the trick, the magician assembles the full deck with two aces on top and two aces on the bottom. Upon getting the combination, *abc*, from the volunteer, the magician deals out *a* cards and then gives the remainder of the deck to the volunteer to count *b* cards off the top. While the volunteer is doing this, the magician moves the bottom card of the first packet of *a* cards to the top that packet. After the volunteer gives back the deck, the volunteer puts one of the packets on top of the other. While this is occurring, the magician removes *a* cards from the bottom of the original deck and adjusts the packet so that an ace is on top and bottom. He then sets the remainder of the original deck aside. The magician places this second *a*-sized packet on the opposite side of the reassembled deck from the other *a*-sized packet. Thus, the aces are in positions 1, *a*, $a+b+1$, and $2a+b$ in a deck of size $2a+b$. Now the volunteer will COAT $a+b$ cards, *c* times. Since clearly $a+b \geq \frac{2a+b}{2}$, the Special 4-Cycle Principle described below, with $n=2a+b$ and $k=a+b$, shows that each of these 4 positions will contain an ace, after any number of COATs.

Next the magician puts the cards where he and the audience cannot see them and turns the top and bottom cards over, thus turning two aces the opposite direction of the rest of the deck. The magician then COATs the deck with $a+b$ cards and flips the top card; thus a third ace is the opposite direction. After one more COAT, the magician again turns over the top card and thus all aces are facing the opposite direction. Finally, the magician shuffles the deck to disguise how the cards were flipped and presents a deck in which all of the aces are facing the opposite way from the rest of the deck.

Special 4-Cycle Principle: Consider a deck of *n* cards. If $k \geq \frac{n}{2}$, then under a sequence of four COATs of *k* cards, the top card (which starts in position 1) orbits through positions *n*, $n-k$, and $k+1$, in turn, before returning to the top of the deck. Consequently, the cards originally in positions *n*, $n-k$, and $k+1$ also cycle through these positions (and position 1) before returning to their original locations.

Proof:

Let the deck have initial ordering $a_1, a_2, \ldots, a_n$. We refer to the proof of the Save at Least 50% Principle for the ordering of the deck after successive COATs. After one COAT, the ordering of the deck is

$$a_{k+1}, a_{k+2}, \ldots, a_n, a_k, a_{k-1}, \ldots, a_1$$

and a_1 is in position *n*. Again from the Save at Least 50% Principle's proof, after two COATs the ordering of the deck is

$$a_{n-k}, a_{n-k-1}, \ldots, a_1, a_{(n+1)-k}, a_{(n+2)-k}, \ldots, a_k, a_n, a_{n-1}, \ldots, a_{k+1}$$

and a_1 is in position $n-k$. Next, after three COATs the ordering of the deck is

$$a_n, a_{n-1}, \ldots, a_{k+1}, a_k, \ldots, a_{(n+1)-k}, a_1, a_2, \ldots, a_{n-k}.$$

Since $\{a_1, a_2, \ldots, a_{n-k}\}$ is $n-k$ cards, $\{a_n, a_{n-1}, \ldots, a_{k+1}, a_k, \ldots, a_{(n+1)-k}\}$ must be *k* cards. Therefore, a_1 must be in position $k+1$ and clearly one more COAT puts a_1 in position 1.■

II. Ditch the Dud

This trick is exactly "Ditch the Dud" as presented by Mulcahy, and utilizes the game of poker (72).

Description of the Trick

"Ask for a spectator who likes poker, as you shuffle the deck. Have ten cards dealt out into a face-down pile, and have that pile further mixed. Pick up the cards and glance at their faces briefly, remarking on how random they are, and yet how they may result in two interesting poker hands. Announce which of you will win. Deal the cards into a face-down row, and alternate with the poker fan in taking cards from one end of the row or the other, until you both have five cards. Compare and see who has the winning poker hand. Your earlier prediction turns out to be correct."

Mathematical Analysis and Trick Explanation

This trick relies on knowing that the ten cards dealt from the top of the deck contain three distinct sets of three of a kind, along with one non-matching "Jonah" card. Therefore, the magician must guarantee that the original shuffling of the large deck keeps this set intact (note this is a set, not an ordered set, and therefore order need not be preserved). Upon the removal of the ten cards from the full deck, they may be legitimately mixed, again because this is a non-ordered set. Based on the Jonah Card principle below, the magician will know that the person whose hand contains the Jonah card will lose.

The magician must therefore be able to guarantee which hand has this card. When showing the audience that the cards are random and will make interesting poker hands, the magician glances to see where the Jonah card is. Therefore, when dealing the cards face down in a line, the magician knows which of these cards is the Jonah card.

The magician, by choosing first, can also determine which cards are in each hand. As Mulcahy points out with his Position Parity property, if only the two cards on the ends can be selected, then by always choosing the card next to the card chosen by the volunteer, the magician is guaranteed to take all of the cards in even positions, or all of the cards in odd positions, depending on the position of the initial card taken. If the magician selects card one first, and then follows the strategy above, he gets all odd positioned cards. Similarly, if he selects card 10 first, then the magician gets all of the even positioned cards. Hence, if the Jonah card is in an odd position, the magician can make sure that the hand with all of the odd cards is the hand he predicted to lose.

Jonah Card Principle: If ten cards consisting of three sets of three of a kind and one non-matching card (a card that forms no pairs with the other cards) are divided into two poker hands, then whoever has the non-matching card loses, without fail. The non-matching card is called the "Jonah" card.

Proof: Let ten cards consisting of three sets of three of a kind and one Jonah card be randomly split into two poker hands of five cards each. By examining the two hands, it is clear that one must have the Jonah card. This "Jonah hand" will also have four of the remaining nine cards. Note that these are four cards chosen from a set of three matching triples, and thus:

a) The best Jonah hand contains a three of a kind and another card in addition to the Jonah card. This leaves the other hand with three of a kind and a pair, so the Jonah hand loses.
b) The second best Jonah hand contains two pairs. This leaves the other hand with a three of a kind, so the Jonah hand loses.
c) The next best Jonah hand contains one pair, plus two mismatched cards in addition to the Jonah card. This leaves the other hand with two pair, so the Jonah hand loses.

Since the four non Jonah cards in the Jonah hand are chosen from a set of three matching triples, the Pigeonhole Principle tells us that the Jonah hand has at least two matching cards. Thus (c) is the worst hand the Jonah hand can have. Hence, all possible hands are accounted for and the Jonah hand will always lose. ■

III. Set Sums

The tricks in this section use special sets whose sums help the magician recognize the identity of specific cards.

III.a. Little Fibs. This trick is exactly "Little Fibs" presented by Mulcahy (89).

Description of the Trick

"Give the deck several shuffles, then deal six cards face down to the table, setting the rest aside. Turn away, requesting that those six cards be thoroughly mixed up. Have any two cards selected by two spectators, who then compute and report the total of the two card values. From that information alone, you promptly name [the number and suit of] each card."

Mathematical Analysis

This trick, like the trick in the last section, relies on a packet of known cards that appear to be randomly shuffled to the top. Therefore, this trick requires some fake shuffling. Once the magician "shuffles," the desired set of cards should be on top. The values of these six cards should form a set of 2-summers as defined below. In order to help the magician remember the suit of each card, he puts the cards in CHaSeD order along with numerical order in the trick's preparation.

A set of 2-summers is a set S where for every $a, b, x, y \in S$ such that $a \neq b$ and $x \neq y$, if $a + b = x + y$, then either $a = x$ and $b = y$, or $b = x$ and $a = y$. So, for example, $\{1, 2, 3, 5\}$ is a set of 2-summers, but $\{1, 2, 3, 4\}$ is not since $1 + 4 = 2 + 3$. Given a set of 2-summers, we can enlarge it using the following lemma.

Lemma: If $B=\{b_1,b_2,\ldots,b_z\}$ is a set of 2-summers such that $b_y<b_{y+1}$ for every integer y with $1\le y\le z-1$, then for any b_{z+1} with $b_{z+1}>b_{z-1}+b_z-b_1$, $B'=B\cup\{b_{z+1}\}$ is a set of 2-summers.

Proof: Let $B=\{b_1,b_2,\ldots,b_z\}$ be a set of 2-summers such that for every $1\le y\le z-1$, $b_y<b_{y+1}$. Let $b_{z+1}\in\mathbb{N}$. In order for $B'=B\cup\{b_{z+1}\}$ to be a set of 2-summers, the sums b_1+b_{z+1}, b_2+b_{z+1}, $\ldots$, b_z+b_{z+1} must be distinct and different from the sums of any other two distinct elements of *B*. Since $b_1<b_i$ for $1<i\le z$, it suffices that $b_{z+1}+b_1>b_{z-1}+b_z$. Thus, if $b_{z+1}>b_{z-1}+b_z-b_1$, it follows that $\{b_1,b_2,\ldots,b_{z+1}\}$ is a set of 2-summers.■

We show below that the set of Fibonacci Numbers is a set of 2-summers. So any set of cards with values equal to distinct Fibonacci numbers will work for this trick.

Fibonacci numbers as a set of 2-summers: Let *F* be the set of Fibonacci numbers: $F=\{1,2,3,5,8,13,\ldots\}$. Then any subset of *F* is a set of 2-summers.

Proof: The Fibonacci numbers are defined inductively by $f_1=1$, $f_2=1$, and $f_{n+1}=f_n+f_{n-1}$ for $n\ge 2$ and, therefore, as a set of distinct integers, $F=\{f_i \mid i\ge 2\}$.

Note that $\{f_2,f_3\}=\{1,2\}$ is a set of 2-summers.

Now assume $\{f_2,f_3,\ldots,f_n\}$ is a set of 2-summers. Then, to show that $\{f_2,f_3,\ldots,f_{n+1}\}$ is a set of 2-summers, the lemma tells us that it's sufficient to show that $f_{n+1}>f_{n-1}+f_n-f_2$. Since the recurrence relation for the Fibonacci numbers is $f_{k+1}=f_{k-1}+f_k$, $f_{n+1}-f_2=f_{n-1}+f_n-f_2$. Thus, $f_{n+1}>f_{n-1}+f_n-f_2$, and $\{f_2,f_3,\ldots,f_n,f_{n+1}\}$ is a set of 2-summers. So, by induction, *F* is a set of 2-summers. It is clear that any subset of a set of 2-summers is also a set of 2-summers. Thus, any subset of *F* is a set of 2-summers.■

Therefore, if the set of cards given to the spectators is a set of Fibonacci numbers, then the spectators can choose any two cards and the magician can identify these cards based on their sum. For example, if the spectator chooses two cards and reports the sum of 10, then the magician knows that only 8 and 2 can make this sum. Hence, the magician knows the cards used were a 2 and an 8. Using a CHaSeD ordering, the magician can even easily memorize the suits of the cards and report this as well. In the case of the first six CHaSeD Fibonacci numbers, the magician would reveal that the volunteer's cards were 2♥ and 8♣.

III.b. Consolidating Your Cards. This trick is a variation of the trick "Consolidating Your Cards" by Mulcahy (93-4).

Description of the Trick

After shuffling, deal out six cards face down from the top of the deck. Tell the volunteer that once you turn away, she is to select three cards from these six, which will be used to determine her credit rating. After she selects her cards, tell her to add the values of the cards together, with red cards as negative values and black cards as positive. Once she reports the sum, the magician either reveals that two cards cancel and gives the suit of the remaining card or gives the value and suit of all three cards.

Mathematical Analysis

This trick again relies on a packet of known cards that appear to be randomly shuffled to the top. Therefore, after the magician performs some fake shuffling, the desired set of cards should be on top. This set, in any order, consists of 9♣, 3♥, A♠, A♦, 3♣, and 9♥ (note that these cards are listed here in CHaSeD order for easy recollection). This packet is important because, following the convention of red as negative and black as positive, it is $\{3^2, -3^1, 3^0, -3^0, 3^1, -3^2\}$.

This allows for the use of the following theorem.

Balanced Ternary Principle: Every integer can be written as a sum of distinct signed powers of 3, and this representation is unique apart from cancelations (i.e., each integer has a unique balanced ternary representation, where 0 is the "empty" representation). For example, $2 = 3^1 - 3^0$ and $13 = 3^2 + 3^1 + 3^0$.

Proof: Assume $k \in \mathbb{Z}^+$.

We induct on *k*. First, note that $1 = 3^0$ and it should be clear that there is no other balanced ternary representation for 1, so 1 has a unique balanced ternary representation. Assume that for every $t \in \mathbb{Z}^+$ with $t < k$, *t* has a unique balanced ternary representation. Let $n \in \mathbb{Z}^+ \cup \{0\}$ such that $3^n \le k$ and $3^{n+1} > k$. Then, by the division algorithm, $k = 3^n \cdot q + r$ for some unique integers $q \in \{1, 2\}$ and $0 \le r < 3^n$.

If $r = 0$ and $q = 1$, $k = 3^n$ is a unique ternary representation. If $r = 0$ and $q = 2$ then $k = 3^{n+1} - 3^n$, a unique balanced ternary representation.

Now assume $r > 0$. Since $r < k$, *r* has a balanced ternary representation by the induction hypothesis. Also, since $r < 3^n$, the largest power of 3 that could appear in a balanced ternary representation of *r* is 3^n. First we assume this balanced ternary representation of *r* does not have a 3^n term. If $q = 1$, then 3^n plus this representation for *r* gives a balanced ternary representation for *k*. If $q = 2$, then $k = 3^{n+1} - 3^n + r$ and *k* has a unique balanced ternary representation.

Next we assume the representation of *r* has a 3^n term. Since $r < 3^n$, the 3^n term must be positive. Also, let $x = r - 3^n$, so the balanced ternary representation for *x* does not have a 3^n

term. Hence, $k = 3^n q + r = 3^n q + 3^n + x = 3^n (q+1) + x$. If $q+1=3$, then $k = 3^{n+1} + 0 \cdot 3^n + x$ provides a balanced ternary representation for *k*, and if $q+1=2$, then $k = 3^{n+1} - 3^n + x$ provides a balanced ternary representation for *k*.

Hence, it is clear that every positive integer has a unique balanced ternary representation.

If $k=0$, then all of the coefficients of powers of 3 are zero and we have the unique "empty" presentation. Finally, if *k* is a negative integer, notice that $k = -|k|$, so we use the balanced ternary representation for $|k|$ to produce the balanced ternary representation for *k*.■

Given this theorem, unless two cards cancel out, the magician finds the balanced ternary representation of the sum reported by the volunteer in order to determine which three cards were used. The magician knows two cards cancelled out if the sum provided is a power of 3, since the unique balanced ternary representation of a power of 3 is simply that number. If this is the case and two cards cancel out, then the magician reveals that two cancelled and reports the value and suit of the remaining card.

For example, if the volunteer reports a sum of 11, then the magician notices that $3^1 + (-3^0) + 3^2 = 11$. So the cards are 3♣, A♦, and 9♣. On the other hand, if the volunteer reports a sum of -9, then the magician notices that $-3^2 = -9$ and therefore two cards must have cancelled out. So he reports that two cards cancelled and the other card is 9♥.

IV. Monotone Subsequences

This section utilizes subsequences of cards which are either constantly increasing or constantly decreasing. Specifically, we use the following result originally proved by Paul Erdős and George Szekeres in 1935 (Gasarch 1):

> In any arrangement of $(k-1)^2+1$ (or more) different numbers, there are always at least *k*, not necessarily beside each other, that are in numerical order. Hence, there is always either a rising run or a falling run of length *k* (or more). (Mulcahy 269)

We provide a proof in the case of $k=3$ as part of our discussion of the next trick, but we do not provide the proof of the general result since it is outside the scope of this project.

IV.a. Five that Jive. This trick is adjusted from "Erdős Numbers" by Mulcahy (274).

Description of the Trick

An accomplice waits where he cannot see or hear the trick as the magician selects a volunteer. After shuffling the deck, the volunteer deals out the top five cards, sets aside the rest of the deck, and shuffles these five cards. The volunteer then lays the cards face up, notes the randomness of the cards, and turns the cards face down once more. After the accomplice enters the room, the magician reveals two cards and the accomplice announces the identity of the remaining three cards (both number and suit).

Mathematical Analysis

This trick relies on a packet of known cards that appear to be randomly shuffled to the top. Therefore, this trick requires some fake shuffling to ensure that the necessary packet of cards is at the top of the deck. This packet should be five cards which both the magician and accomplice have memorized. The magician and accomplice will have previously agreed on a linear order relation on the cards in the packet, so that for any two cards, one is defined to be 'greater' than the other. Since this is the $k = 3$ case of the Erdős-Szekeres result, there is either an increasing or decreasing subsequence of length three. While the cards are face up, the magician locates the monotone subsequence. After the volunteer flips all of the cards face down, the magician flips the cards that are *not* in the monotone subsequence face up with the accomplice present. The magician reveals cards from right to left to indicate an increasing sequence, and reveals cards from left to right to indicate a decreasing sequence. The following theorem proves this $k = 3$ case, which guarantees that the magician need only reveal two cards in this manner for the accomplice to announce the identities of the three cards that remain face down.

Special Case of Erdős-Szekeres: For any sequence of five distinct numbers, there is always a monotone subsequence of length three.

Proof: Let $\{a, b, c, d, e\}$ be a sequence of distinct numbers.

Assume $a < b$. If $b < c$, then $\{a, b, c\}$ is an increasing subsequence. Similarly, if $b < d$ or $b < e$, then $\{a, b, d\}$ or $\{a, b, e\}$ is an increasing subsequence, respectively. Otherwise, $b > \max\{c, d, e\}$. If $c > d$, then $\{b, c, d\}$ is a decreasing subsequence. Similarly, if $c > e$, then $\{b, c, e\}$ is a decreasing subsequence. Otherwise, $c < \min\{d, e\}$. If $d > e$, then $\{b, d, e\}$ is a decreasing subsequence. The only remaining option is if $d < e$, in which case $\{c, d, e\}$ is an increasing subsequence.

Now assume $a > b$. If $b > c$, then $\{a, b, c\}$ is a decreasing subsequence. Similarly, if $b > d$ or $b > e$, then $\{a, b, d\}$ or $\{a, b, e\}$ is a decreasing subsequence, respectively. Otherwise

$b < \min\{c,d,e\}$. If $c < d$, then $\{b,c,d\}$ is an increasing subsequence. Similarly, if $c < e$, then $\{b,c,e\}$ is an increasing subsequence. Otherwise, $c > \max\{d,e\}$. If $d > e$, then $\{c,d,e\}$ is a decreasing subsequence. The only remaining option is if $d < e$, in which case $\{b,d,e\}$ is an increasing subsequence.

Thus, in every case the sequence $\{a,b,c,d,e\}$ has a monotone subsequence of length three.■

IV.b. Ten Soldiers. This trick is adjusted from "Ten Soldiers" by Mulcahy (264).

Description of the Trick

This trick has the same procedure as the previous trick, Five that Jive, with the exceptions that ten cards are taken from the top of the deck instead of five, and four cards are left to be revealed by the accomplice instead of three.

Mathematical Analysis

This trick applies the $k = 10$ case of the Erdős-Szekeres result. The proof of this case, like the general result, is outside the scope of this project.

IV.c. Clear Cut Diamonds. This trick is adjusted from "Slippery Enough" presented by Mulcahy (271).

Description of the Trick

The magician selects a volunteer from the audience and gives him the deck to remove the diamonds and place them face-up in a line, in whatever order he chooses. The magician surveys the row of thirteen cards (possibly asking the audience to do the same), then has the volunteer flip each card face down. At this point, an accomplice who has neither seen nor heard what has happened so far is brought into the room. The magician turns over some of the cards (usually eight) and the accomplice names (correctly) the cards which are missing.

Mathematical Analysis

This trick works because of statistics and simple communication. Since there are thirteen diamonds in a deck of cards, there are $13!$ possible arrangements of the cards in this trick. Of these possible sequences of cards, there is approximately a 98.4% chance that the sequence has a monotone subsequence of length five (Mulcahy 271). If this is the case, the magician uses such a sequence. If not, then the magician uses the $k = 4$ case of the Erdős-Szekeres result, and reveals nine cards instead of eight.

The magician reveals cards in the same manner as in Five that Jive. Since all of the diamonds are on the table, the accomplice can easily determine which cards are face down, and can use the magician's cues to determine their order.

V. Error Correcting Codes

This section deals with the mathematical idea of error correcting codes. These are codes that have a built in mechanism that enables the detection and correction of errors.

V.a. A Horse of a Different Color. This trick is exactly "A Horse of a Different Color" by Mulcahy (288-9).

Description of the Trick

"An audience member is invited to select any three cards from the deck and lay them in a face-up row on the table. You supplement this row with three more face-up cards of your own choosing.

"Say, 'Think of these cards as six horses in a stream.' Before that sinks in, add, 'No doubt you've heard the expression, "Don't change horses in the middle of a stream." Actually, that's exactly what I want you to do. Please change any one horse – for a horse of a different color!'

"The audience member replaces any one of the cards on the table with a new card from the deck, subject to the provision that the new card must not be the same color as the one it replaces. Your accomplice now enters the room, and soon identifies which card on the table was switched."

Mathematical Analysis

This trick uses the concept of error correcting codes. Specifically, it uses a linear binary code. We begin by describing a seemingly easier trick, where the volunteer chooses only two cards. We will then explain how to add a third card from the volunteer without adding any actual complexity to the trick. The code, *C*, is defined by $C:(a,b)\mapsto(a,b,a,b,a+b)$ where

$a,b \in \{0,1\}$ and addition is done modulo 2. The following logic proves that if a received code word is known to contain exactly one error, the recipient can recover the correct code word.

Assume *x* is the incorrect digit and assume that $c = a + b \pmod 2$ for the original *a* and *b*.

1) If the received (corrupted) code is (x,b,a,b,c) or (a,b,x,b,c), then positions one and three do not match and so one of those positions contains the error. Since $a+b=c$ but $x+b \neq c$, the assistant can further pinpoint the location of *x*.
2) If the received code is (a,x,a,b,c) or (a,b,a,x,c), then positions two and four do not match and so one of those positions contains the error. Since $a+b=c$ but $a+x \neq c$, the assistant can further pinpoint the location of *x*.
3) If the received code is (a,b,a,b,x) then the sum of positions one and two, and the sum of positions three and four, do not equal the number in the fifth position, and hence there is an error in position five.

Hence, the location of the error is identifiable given only the corrupted code and the knowledge that the code contains exactly one error. Since the code uses a binary system, by knowing the position of the error, it is simple to correct the error by switching the corrupted digit with the other element of $\{0,1\}$.

Given the assumption that black cards have value zero and red cards have value one, the magician uses the first two cards chosen by the volunteer to choose three additional cards. The volunteer then changes one card's color (i.e., its value), thus introducing an error to the code word. By applying the logic presented above, the accomplice can then determine which card was switched.

Recall that in the description of this trick, the volunteer chooses three cards, not two. This extra card is ignored in the magician's choice of cards and he simply changes the code to $C' : (a,b,z) \mapsto (a,b,z,a,b,a+b)$ where $a,b,z \in \{0,1\}$. The accomplice performs the same error checking as before, except that if no error is detected in the code, then the error must be in the third position, which is the position not checked by the code. Hence, the accomplice detects and corrects any single error.

V.b. And Now for Something Completely Different. This trick is exactly "And Now for Something Completely Different" from Mulcahy (302).

Description of the Trick

"Give out a deck of cards for shuffling. Take it back, and fan it to reveal that that the cards are all face up. Comment, 'These aren't mixed up very well. Look, they all face the same way!' Split the deck near the middle, and flip over one half, before riffling the two parts together. Perhaps hand the deck out again for additional shuffling. 'That's better,' you conclude, as you fan the cards again to show that they are well and truly mixed now.

"Invite an audience member to select any two cards from the deck and place them side by side on the table. You rapidly supplement these with two cards of your own choosing, to form a row of four cards.

"'Four random cards, some Red, some Black, some face down! And now for something completely different. Please change any one card. For instance, you could just turn one of these cards over, or you could switch a [face up] Red card there for a [face up] Black one from the deck, or vice versa.'

"The audience member does as instructed. Your accomplice now enters the room for the first time, and soon identifies which card on the table was switched. Even better, if the switched card is now face down, she can tell whether it was originally Black or Red. Furthermore, if the switched card is face up, she can tell whether it was originally a different color or face down."

Mathematical Analysis

This trick uses a similar idea as in the previous trick, using a ternary linear code rather than a binary code. The three properties of cards comprising the ternary system will be red (corresponding to -1), black (1) and face down (0). The code takes an ordered pair (a,b), where $a,b \in \{-1,0,1\}$, and turns it into an ordered quadruple $(a,b,a+b,b-a)$, where $a+b$ and $b-a$ are both reduced modulo 3 with the convention of recording 2 as -1 (note that 2 and -1 are equivalent modulo 3).

In a correctly coded message (s,t,u,v), with $s,t,u,v \in \{-1,0,1\}$, $t+u+v=0$, since $b+(b+a)+(b-a)=3b=0(\bmod 3)$. Similarly, since $a+(b+a)=b+2a=b-a(\bmod 3)$, $s+u=v$; $s+v=t$ since $a+(b-a)=b$; and $s+t=u$ trivially. If we know that a received code word contains exactly one error, we can use the facts above to determine the location of the error. Looking at the received message (s,t,u,v), with exactly one error, the following are true:

- The error exists in the first position if and only if $t+u+v=0$.
- The error exists in the second position if and only if $s+u=v$.
- The error exists in the third position if and only if $s+v=t$.
- The error exists in the fourth position if and only if $s+t=u$.

This property shows that if one of the equations is true, the three variables used cannot contain the error and the fourth variable must then contain the error. However, this trick really requires at most one error. If no error is present, all the above equations will be true. Therefore, this trick has two variations, one where exactly one error must occur and one where at most one error must occur.

Once the accomplice determines the location of the error, the accomplice can use the original coding method to determine what the code word should have been, and hence whether the card was originally face down, red or black. For example, if the presented code is $(s,t,u,v)=(1,1,1,-1)$, then the accomplice notices that $s+u=1+1=-1(\text{mod}\,3)=v$ and the error must be in the second position. Since $u=1=1+0=s+t$ (the equation for position three), we know that position two should have contained a zero and, hence the card in that position was originally face down.

VI. Fitch Cheney's Five-Card Twist

This trick is adapted from the trick with the same name presented by Mulcahy (306).

Description of the Trick

The magician selects a volunteer to shuffle the deck and choose any five cards. The magician examines the cards, hides one of the five cards, sets the remaining four cards in a face-up row, and has the volunteer retrieve an accomplice from outside the room. The accomplice briefly examines the cards and identifies the missing card.

Mathematical Analysis

Unlike many of the other tricks presented, this trick involves no fake shuffling or known cards; the accomplice knows nothing about the five cards before entering the room. Clearly the magician is using the four remaining cards to identify the missing card. Hence, we will examine the decision making process for the magician's selected four cards. First, since there are five cards present and only four suits, the Pigeonhole Principle tells us that at least one suit must be used twice. Thus, the magician hides one of the cards from a duplicated suit. In order to communicate the suit to the accomplice, the volunteer calculates the sum of the remaining cards (with jack, queen, king and ace equal to 11, 12, 13, and 1, respectively), then reduces the sum modulo 4 with $0 = 4 \pmod 4$ referring to the fourth position. The magician places the card that determines the suit of the hidden card in the position determined by this sum. The other three cards will communicate the value of the hidden card.

Note that there are 13 cards of each suit. Thus, for any two cards with values *a* and *b*, $|a-b| \le 6$, and so the hidden card and the card that identifies the suit of the hidden card are within 6 of each other. Assuming that $a < b$, the magician hides *a* if $b \in (a, a+6]$ and hides *b* otherwise. For example, in choosing between hiding 3♥ or K♥, the magician hides 3♥, because $13 \notin (3,9]$.

By identifying suits as low to high following the CHaSeD ordering, every card is uniquely higher or lower than any other card (i.e., 3♥ is greater than 10♣). Thus, by applying the following rule, the magician can tell the accomplice what number to add to the value of the visible suit card based on the relative degrees and order of the remaining three cards.

Using L, M, H for "low," "middle," and "high," respectively, where LMH means the remaining three cards are in the order low, medium, high from left to right:

- LMH: Add 1.
- LHM: Add 2.
- MLH: Add 3.
- MHL: Add 4.
- HLM: Add 5.
- HML: Add 6.

Thus, the magician can order the four remaining cards in a way that identifies the hidden card, both in terms of suit and value.

For example, assume the volunteer selects 7♣, 7♦, 8♣, J♥, and Q♠. The magician then notes that two clubs are present (the 7 and 8) and that $8 \in (7,13]$. Hence, the 8 is hidden. Given the values of the remaining cards, the magician determines $7+7+11+12=37=1(\bmod 4)$ and puts 7♣ in the first position to communicate the suit of the hidden card. Since 8 is one card after 7, the magician needs to communicate that 1 must be added. Thus, the order of the remaining three cards is low, medium, high. Since hearts are less than spades, which are less than diamons, when the accomplice walks in he will see 7♣, J♥, Q♠, 7♦ and, in that order. The accomplice concludes that the hidden card is 8♣.

Conclusion and Personal Reflections

Over the course of this examination of the mathematical bases of various card tricks, it became clear to me that these tricks use a wide range of mathematics. The concepts for each trick varied widely, from the combinatorics used in COATs and the discrete mathematics of the Pigeonhole Principle, to error detecting and correcting codes. This demonstrated to me that mathematics is a wide ranging field, even in applications as seemingly simple as card tricks. My investigation raises the question of how other areas of mathematics may lead to new tricks. Thus, my further research into this field might focus on inventing new tricks based on mathematics, whether simple or advanced, that I did not examine in this paper. Additionally, certain tricks in Mulcahy's book seemed interesting mathematically, but I excluded them from the paper due to their confusing nature and a lack of consistent performance success. For example, "Lucky Number One and Thirteen" requires either more work or an alteration to make it a more easily accomplished illusion (Mulcahy 144-5). Additionally, in the future I would like to examine other sources of tricks such as *Magical Mathematics* by Persi Diaconis and Ron Graham, as well as various works of Martin Gardner.

The content of this paper is well suited for demonstrating how mathematics can be fun. One possible way I could spread this message is by using this paper as a basis for one or more presentations in a high school mathematics class. Since many of the concepts of the paper are accessible to high school students, the presentation of these tricks in such an environment is practical. I would present the tricks, followed by their explanation in mathematical terms. This activity would both promote mathematics as enjoyable and introduce high school students to mathematics based on proofs, not just calculations.

Another direction for my future investigation is tricks using other types of cards. All of the tricks in this paper use a standard deck of playing cards, which have three characteristics: suit, color, and number. Since the suit determines the color, one can argue that each card has just two characteristics. It may be rewarding to investigate possible tricks involving more complicated decks, such as the cards used in the game Set. These cards each have four attributes rather than three.

Overall, card tricks are interesting mathematically as well as being interesting to lay people. Their study led me to a better understanding of how math can be applied to seemingly unrelated fields in addition to the educational benefits of demonstrating mathematics in such a field to stimulate interest. I gained a deeper understanding of proof as a result of this study, moving away from proving typical mathematical results to proving card tricks. Through such an examination, I more fully understand the idea of proof.

Bibliography

Bona, Miklos. *Introduction To Enumerative Combinatorics*. Boston: McGraw-Hill Higher Education, 2005. Print.

Gasarch, William. "Five Proofs of the Subsequence Theorem." (n.d.): n. pag. University of Maryland. Web. 8 Feb. 2015. <http://www.cs.umd.edu/~gasarch/BLOGPAPERS/subseq4proofs.pdf>.

Gilbert, Jimmie, and Linda Gilbert. *Elements of Modern Algebra*. 7th ed. Pacific Grove, CA: Brooks/Cole, 2000. Print.

Knuth, Donald Ervin, Tracy Larrabee, and Paul M. Roberts. *Mathematical Writing*. Washington, D.C.: Mathematical Association of America, 1989. Print.

Mulcahy, Colm Kevin. *Mathematical Card Magic: Fifty-two New Effects*. Boca Raton: CPC, 2013. Print.

Merzback, Uta C., and Carl B. Boyer. *A History of Mathematics*. Third ed. Hoboken: Wiley, 2011. Print.

Richmond, Bettina, and Thomas Richmond. *A Discrete Transition to Advanced Mathematics*. Belmont, CA: Thomson/Brooks/Cole, 2004. Print.

Rose, H. E. *Linear Algebra: A Pure Mathematical Approach*. Basel: Birkhäuser Verlag, 2002. Print.